

Weak Omega Categories I

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Abstract

We develop a theory of weak omega categories that will be accessible to anyone who is familiar with the language of categories and functors and who has encountered the definition of a strict 2-category.

The most remarkable feature of this theory is its simplicity. We build upon an idea due to Jacques Penon by defining a weak omega category to be a span of omega magmas with certain properties. (An omega magma is a reflexive, globular set with a system of partially defined, binary composition operations which respects the globular structure.)

Categories, bicategories, strict omega categories and Penon's weak omega categories are all instances of our weak omega categories. We offer a heuristic argument to justify the claim that Batanin's weak omega categories also fit into our framework.

We show that the Baez-Dolan stabilization hypothesis is a direct consequence of our definition of weak omega categories.

We define a natural notion of a pseudo-functor between weak omega categories and show that it includes the classical notion of a homomorphism between bicategories. In any weak omega category the operation of composition with a fixed 1-cell defines such a pseudo-functor.

Finally, we define a notion of weak equivalence between weak omega categories which generalizes the standard definition of an equivalence between ordinary categories.

0.1 Introduction

This paper begins the development of a theory of weak, higher dimensional categories which parallels closely the familiar theory of ordinary (1

dimensional) categories [15]. It should be accessible to anyone comfortable with the language of categories and functors. In particular, concepts such as operads and monads play no role in the basic definitions. In the sequel, Part II, we shall build upon the foundation laid down in Part I to develop some of the more technical aspects of the theory: a notion of weighted limits and the construction of the weak omega category of small weak omega categories. In Part II we shall also construct a functor from any Quillen model category to this omega category of small omega categories and construct a weak omega category whose n-dimensional morphisms are n-dimensional cobordisms with corners.

A category can be defined as a directed graph with a partially defined, binary composition law that satisfies additional axioms: associativity of composition and the existence of right and left identity for each vertex (or object). Our weak omega categories are defined in a similar spirit.

We start with an omega graph (a reflexive, globular set) together with a system of partially defined, binary composition laws that respects the graph structure. Such an omega graph together with its system of composition laws is called an omega magma.

An omega magma is a strict omega category if its composition laws satisfy the higher dimensional generalizations of the associative, identity and interchange laws exactly, i.e. these axioms hold as equations between elements of the omega magma. A strict omega category all of whose cells above dimension 1 are identities is just an ordinary category. The standard example of a strict 2-category [2,7](all cells above dimension 2 are identities) is the 2-category whose objects are small categories, whose morphisms are functors and whose 2-dimensional arrows are natural transformations between functors.

An omega magma is a weak omega category if its composition laws satisfy the higher dimensional generalizations of the associative, identity and interchange laws in a “relaxed” way: the laws are required to hold only “up to an equivalence”. An equivalence in a weak omega category is a higher dimensional arrow in the category that behaves like a homotopy equivalence in homotopy theory. It is generally something less than an isomorphism but still preserves enough structure to behave like one for categorical purposes. The simplest example is already familiar to the reader: any functor inducing a categorical equivalence between a pair of ordinary categories is a 1-dimensional equivalence in this more general sense (at least if we admit the axiom of choice); such a functor need not be an isomorphism of categories.

Any bicategory [6,7,12,14,15,16] is a weak 2-category. Monoidal categories [8,15] are bicategories having only one object and are the most familiar instances of weak 2-categories.

The crux of any theory of weak omega categories is its method for expressing mathematically the coherence conditions that assert that the desired categorical laws hold “up to equivalence”. The device we have chosen for this purpose first appeared implicitly in the work of Jacques Penon [18]. It traces its roots to the standard coherence theorem for bicategories [12,14]. The latter result asserts that any bicategory can be embedded in a strict 2-category via a functor which preserves the 1-dimensional composition law only up to isomorphism. Penon’s wonderful idea was simply to turn the conclusion of this coherence theorem into a definition.

Of course some subtlety must be involved here. Category theorists have known for some time [11] that there are weak 3-categories which cannot be embedded into a strict 3-category. Thus it is not possible to define a weak omega category as an omega magma that admits an appropriate, structure preserving embedding into a strict omega category.

Penon circumvented this problem by redefining a morphism from one omega magma to another to be a span of omega magma homomorphisms, i.e. a diagram of the form $X \leftarrow Z \rightarrow Y$ in the category of omega magmas. We call this a span from X to Y and call X the domain and Y the codomain of the span. One can then define a weak omega category to be an omega magma that is the domain of a span to a strict omega category. To avoid a trivial theory one must of course impose some additional conditions on this span.

We should warn the reader that he will not find this informal explanation anywhere in Penon’s paper [18]. Nonetheless it lies just beneath the surface of his work and soon becomes apparent once one attempts to unravel his definition of weak omega categories (which he called “prolixes”).

The approach to weak omega categories taken here differs in several other ways from Penon’s.

We highlight the idea that a weak omega category is first of all an omega magma: an omega graph with a system of partially defined, binary composition operations. The coherence conditions that make the omega magma a weak omega category are expressed by a particular span from the omega magma to a strict omega category. This span must satisfy certain simple axioms. In Penon’s theory omega magmas are incidental, simply stepping stones on the way to the construction of a monad on the category of reflexive, globular sets whose algebras are his weak omega categories.

We might add that our emphasis on omega magmas and their systems of binary compositions also distinguishes this work from theories of higher dimensional categories that define them as algebras for higher

dimensional operads [3,5,9,13,14,17].

A much more important difference is our willingness to entirely remove certain restrictions Penon places upon the spans which define his weak omega categories . If $X \leftarrow Z \rightarrow Y$ is a span defining X as a weak omega category, Penon requires that the strict omega category Y be freely generated by the omega graph underlying X . In addition the omega magma Z is required to be freely constructed (via an adjunction) from the free omega magma generated by the same omega graph, viz. the one underlying X . (The omega magma Z Penon calls a "stretching" of the strict omega category Y .) These last restrictions are inherent in Penon's construction of the monad whose algebras are his weak omega categories.

It soon became apparent to us that these requirements render Penon's theory inflexible, making constructions and proofs hard and obscure in situations where they should be easy and transparent. We therefore have chosen to allow the strict omega category Y and the omega magma Z to be completely arbitrary while retaining certain axioms on the span from X to Y . The price we pay for this is that our weak omega categories are no longer the algebras for some suitable monad on omega graphs. Still, the benefits of this generality far outweigh its costs.

The biggest benefit lies in the great simplicity of the resulting theory. This allows us to explore territory as yet inaccessible to other theories of weak omega categories [3,5,9,13,14,17,18,19] (see the next subsection for details).

Many mathematicians have offered encouragement, insight and patient answers to our often benighted questions during the course of this research. For their generous assistance we would like to thank John Baez, Michael Batanin, Clemens Berger, Ronnie Brown, Eugenia Cheng, John Duskin, Paul Goerss, Peter Johnstone, G. Max Kelly, Steven Lack, Tom Leinster, Saunders MacLane, Peter May, Tim Porter, Charles Rezk, Steven Schnauel, Ross Street, Earl Taft, Mark Weber, Noson Yanofsky and David Yetter.

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0.2 Contents of this paper

Section 1 begins with the definitions of globular sets, omega-graphs, omega magmas and strict omega categories. We discuss briefly the concept of a locally presentable category and the notion of an essentially algebraic theory. The category of models of any essentially algebraic theory is a locally presentable category. Locally presentable categories have many convenient properties which facilitate the construction of adjunctions.

Section 2 presents the definition of a weak omega category \mathbf{X} as a span of omega magmas that satisfies certain simple axioms. The domain of such a span is the underlying omega magma of the weak omega category. We introduce the notion of a Penon map between omega magmas. Such maps appear as the codomain “leg” of the span defining a weak omega category and they structure the coherence data for the span. We define an obvious notion of an omega functor between weak omega categories and show that the resulting category, *Omega_Cat*, of small, weak omega categories and omega functors is the category of models of an essentially algebraic theory.

In Section 3 we begin to justify our definition of weak omega categories by showing that *Cat*, the category of small categories and functors, is a retract of the full subcategory of *Omega_Cat* containing those objects which are *1-skeletal*, i.e. those objects whose underlying omega magma (the domain of the defining span) has only identity cells above dimension 1. We also show that *Strict_Cat*, the category of strict omega categories and strict omega functors is in fact isomorphic to a full subcategory of *Omega_Cat*.

We continue justifying our definition in Section 4 by considering bicategories. We show that *Bicat*, the category whose objects are bicategories and whose morphisms are strong homomorphisms between bicategories (preserving operations and identities “on the nose”) is a retract of the full subcategory of *Omega_Cat* whose objects are *2-skeletal*.

Section 5 examines the relationship between two other definitions of weak omega categories and our own. We show that the category of Penon’s weak omega categories and omega functors, *Prolixe*, is a retract of a certain full subcategory of our *Omega_Cat*. We also offer an informal argument to support our contention that *Batanin_Cat*, the category whose objects are Batanin’s weak omega categories (i.e. the algebras for the initial contractible, higher dimensional operad with a system of compositions) are instances of our weak omega categories.

In Section 6 we point out one immediate consequence of our definition. For each $k \geq 1$ one can define a full subcategory $nCat_k$ of *Omega_Cat* in which an object is a weak omega category whose three

defining omega magmas each has only a single cell in every dimension $\leq k - 1$. Objects of $nCat_k$ are called k -tuply monoidal weak $(n + k)$ -categories by Baez and Dolan [4]. We define an obvious pair of functors in opposite directions connecting $nCat_k$ and $nCat_{k+1}$ whenever $k \geq 2$ and observe that each composite functor is the identity functor. These functors simply shift all the data defining weak omega categories and omega functors by a single dimension. In this sense the Baez-Dolan stabilization conjecture [4] is a simple consequence of our definition of weak omega category.

Section 7 discusses several methods for constructing weak omega categories from given ones and for recognizing that a given omega magma can be given the structure of a weak omega category.

$\Omega\text{-Cat}$ is the category of models of an essentially algebraic theory. As such it is locally (finitely) presentable and thus complete and cocomplete with respect to ordinary conical limits and colimits. We show that the left adjoint half of Penon's monad [18] can be used to define a functor from $\Omega\text{-Graph}$ to $\Omega\text{-Cat}$ and thus gives a way of functorially associating with any omega graph a weak omega category. Moreover, Penon's methods also show that to any morphism from an omega graph to a strict omega category one can associate functorially a weak omega category whose defining span has as its codomain the given strict omega category.

If \mathbf{X} is a weak omega category and a, b a pair of cells of dimension $i - 1 \geq 0$ we construct the weak omega category $\mathbf{X}(a, b)$ whose objects are i cells of \mathbf{X} with domain a and codomain b .

Section 7 ends with a definition of *categorical equivalence relations* and offers a condition sufficient to guarantee that an omega magma equipped with such a relation is the underlying magma of a weak omega category.

In Section 8 we define a natural notion of omega pseudo-functor between weak omega categories. Every omega functor is an omega pseudo-functor. We show that omega pseudo-functors compose in the obvious way and that therefore there is an ordinary category $PF\text{-}\Omega\text{-Cat}$ whose objects are weak omega categories and whose morphisms are omega pseudo-functors. If \mathbf{X} is a weak omega category we show that the operation of composition with a fixed object of $\mathbf{X}(b, c)$ is a pseudo-functor from $\mathbf{X}(a, b)$ to $\mathbf{X}(a, c)$ for any triple of $i - 1$ cells a, b, c .

We define the notion of a *proper* homomorphism between classical bicategories and show that any proper homomorphism between bicategories is an omega pseudo-functor. We know of no examples of classical homomorphisms that are not proper but in principle such examples could exist. We also show that any omega pseudo-functor between 2-skeletal,

weak omega categories is a proper homomorphism between classical bi-categories.

We conclude in Section 9 by defining the notions of *weak equivalence* and of *omega equivalence* between weak omega categories. These definitions are the foundation of a theory of weighted limits which shall be developed in part II of this work. The basic idea is to define a suitable “components” functor, Π , from a certain category *Tame_Omega_Cat* to *Set*, the category of small sets. The category *Tame_Omega_Cat* has as its objects small, weak omega categories and as its morphisms those omega pseudo functors which we shall call *tame*. An omega pseudo-functor is tame when it preserves a class of arrows we call internal equivalences, much as a functor preserves isomorphisms and as a homomorphism between bicategories preserves 1-cell equivalences. The class of tame omega pseudo-functors includes all omega functors between weak omega categories and all omega pseudo-functors between weak n-categories for n finite.

We say that an omega pseudo-functor $\mathbf{F} : \mathbf{X} \rightarrow \mathbf{Y}$ is a weak equivalence if $\Pi(\mathbf{F})$ is an isomorphism and if $\Pi(\mathbf{F}(a, b))$ is also an isomorphism for all pairs of $i - 1$ cells a, b where

$$\mathbf{F}(a, b) : \mathbf{X}(a, b) \rightarrow \mathbf{Y}(Fa, Fb)$$

is the restriction of \mathbf{F} to $\mathbf{X}(a, b)$. It is easily seen that two weak omega categories which are 1-skeletal (and hence are ordinary categories) are weakly equivalent in this sense if and only if they are equivalent as ordinary categories and the appropriately modified assertion also holds for 2-skeletal, weak omega categories..

1 Recollections

1.1 Omega magmas and strict omega categories

We begin by recalling some definitions which are essential to our theory of higher dimensional categories. These are all standard and can be found for example in [5,14] (with some differences in notation). The only one which may be unfamiliar is the notion of an omega magma (compare [14,18]). An omega magma can be understood as an object that would be a strict omega category but for the failure of its composition laws to satisfy the associative, interchange and identity laws which must hold in any strict omega category.

Definition 1 A *globular set* X is a sequence of sets $(X_i, i \geq 0)$ together with functions $dom_{i-1}^i, cod_{i-1}^i : X_i \rightarrow X_{i-1}$ defined for $i > 0$ that

satisfy the so-called **globular relations**:

$$\begin{aligned} \text{dom}_{i-1}^i \circ \text{dom}_i^{i+1} &= \text{dom}_{i-1}^i \circ \text{cod}_i^{i+1} \\ \text{cod}_{i-1}^i \circ \text{dom}_i^{i+1} &= \text{cod}_{i-1}^i \circ \text{cod}_i^{i+1} \end{aligned}$$

for all integers $i > 0$

The elements of the set X_i are called the *cells* of X of dimension i . The function dom_{i-1}^i assigns to each i -cell its domain and the function cod_{i-1}^i assigns to each i -cell its codomain. The terminology is intended to evoke the idea that an i -cell of X is a kind of “ i -dimensional morphism” whose domain and codomain are both “ $(i-1)$ -dimensional morphisms”.

For $j - i > 0$ we denote the $(j - i)$ -fold composite

$$\text{dom}_i^{i+1} \circ \text{dom}_{i+1}^{i+2} \circ \dots \circ \text{dom}_{j-1}^j$$

by dom_i^j .

There is a category, Glob_Set , in which an object is a small globular set and in which a morphism $F : X \rightarrow Y$ is a sequence of functions $(F_i : X_i \rightarrow Y_i)$ which commute with the domain and codomain functions of X and Y .

Any directed graph G determines a globular set by defining G_0 to be the set of vertices and G_i to be the set of directed edges for $i \geq 1$. For $i > 1$ dom_{i-1}^i is the identity map and for $i = 1$ this function assigns to a directed edge the source vertex for that edge. For $i > 1$ cod_{i-1}^i is also the identity map and for $i = 1$ this function assigns to a directed edge its target vertex. Thus to any category there is a globular set determined by the category’s underlying directed graph whose vertices are the objects of the category and whose directed edges are the morphisms.

The directed graph associated with a category has additional structure, the directed edges corresponding to the identity morphisms, that isn’t part of the definition of a globular set. Globular sets with this additional structure we shall call *omega graphs*.

Definition 2 An **omega graph** X is a globular set together with a sequence of **identity functions** $\text{id}_i^{i-1} : X_{i-1} \rightarrow X_i$ for $i > 0$ which satisfy the following relations:

$$\begin{aligned} \text{dom}_{i-1}^i \circ \text{id}_i^{i-1}(x) &= x \\ \text{cod}_{i-1}^i \circ \text{id}_i^{i-1}(x) &= x \end{aligned}$$

for all $x \in X_{i-1}$. An i -cell of X in the image of id_i^{i-1} is called an **identity cell** of X . The $(j - i)$ -fold iterated composite id_j^i of identity functions is defined by the equation

$$\text{id}_j^i = \text{id}_{i+1}^i \circ \text{id}_{i+2}^{i+1} \circ \dots \circ \text{id}_j^{j-1}$$

In the literature one often encounters omega graphs under a different name: reflexive, globular sets. We prefer our own terminology both because of its economy and because its use avoids the risk of confusing globular sets with reflexive, globular sets.

There is a category, *Omega_Graph*, in which an object is a small omega graph and a morphism is a morphism of the underlying globular sets which also commutes with the identity cell functions.

We shall often have occasion to consider omega graphs in which all cells above a certain dimension are identity cells.

Definition 3 *An omega graph X is **n -skeletal** if every cell of dimension $> n$ is an identity cell.*

Our theory of weak omega categories is built around the notion of an omega graph with a compatible family of partially defined, binary composition laws.

Definition 4 *An **omega magma** is an omega graph together with a sequence of ternary relations*

$$Comp_i^j \subseteq X_j \times X_j \times X_j, 0 \leq i < j$$

*These relations are called **composition relations** and must satisfy the following axioms:*

1. *if $(a, b, c) \in Comp_i^j$ then $cod_i^j(a) = dom_i^j(b)$*
2. *if $(a, b, c) \in Comp_i^j$ and $(a, b, d) \in Comp_i^j$ then $c = d$*
3. *if $a, b \in X_j$ and $cod_i^j(a) = dom_i^j(b)$ then $\exists c$ and $(a, b, c) \in Comp_i^j$*
4. *if $i + 1 = j$ and $(a, b, c) \in Comp_i^j$ then $dom_i^j(c) = dom_i^j(a)$ and $cod_i^j(c) = cod_i^j(b)$*
5. *if $i + 1 < j$ and $(a, b, c) \in Comp_i^j$ then*

$$(dom_{j-1}^j(a), dom_{j-1}^j(b), dom_{j-1}^j(c)) \in Comp_i^{j-1}$$

and

$$(cod_{j-1}^j(a), cod_{j-1}^j(b), cod_{j-1}^j(c)) \in Comp_i^{j-1}$$

Each of the ternary relations which is part of the structure of an omega magma X is a partially defined, binary composition law. It is usually more convenient to use the “infix” notation to denote composite cells in an omega magma. Thus if $a, b \in X_j$ and $cod_i^j(a) = dom_i^j(b)$ we

shall write $a \circ_i^j b$ to denote the unique j -cell c for which $(a, b, c) \in \text{Comp}_i^j$. Moreover, any appearance of the notation $a \circ_i^j b$ will be taken to imply that $a, b \in X_j$ and $\text{cod}_i^j(a) = \text{dom}_i^j(b)$. This convention will save much tedious repetition of obvious hypotheses. The partial operation \circ_i^j will sometimes be referred to as composition of j -cells over i -cells.

A category C can be thought of as a 1-skeletal omega magma whose only non trivial composition law is the relation Comp_0^1 . If $(f, g, h) \in \text{Comp}_0^1$ then the codomain of the morphism f coincides with the domain of the morphism g and the composite $g \circ f$ is equal to h .

Here we point out a potential source of confusion. In any category it is conventional to write the composite morphism of the diagram

$$\begin{array}{ccc} & f & g \\ X & \rightarrow & Y \rightarrow Z \end{array}$$

as $g \circ f$. On the other hand, if we are thinking of the category as a 1-skeletal omega magma the same element will be written as $f \circ_i^j g$. Note the reversal of order!

There is a category, *Omega_Magma*, in which an object is a small omega magma and a morphism is a morphism of underlying omega graphs which commutes with the partially defined composition operations.

Omega magmas can be quite complicated objects since their composition laws satisfy no axioms aside from those which ensure compatibility with the omega graph structure. However a consideration of low dimensional examples helps to give some insight into the significance of their defining axioms and shows how omega magmas differ from categories.

As a simple first example let's consider a 1-skeletal omega magma X which has only a single 0-cell denoted by $*$ and which is generated in dimension 1 by a single 1-cell f together with the identity cell $\text{id}_1^0(*)$. Then the set X_1 of 1-cells is just the set of all possible ways of inserting parentheses into a finite sequence consisting of repetitions of the symbols f and $\text{id}_1^0(*)$ so as to represent a meaningful sequence of binary compositions yielding a single cell. The operation \circ_0^1 is just concatenation of parenthesized sequences. It is important to remember that while we call $\text{id}_1^0(*)$ an identity cell the two elements of X_1 denoted by $\text{id}_1^0(*) \circ_0^1 f$ and f (for example) are **distinct**.

Here is a slightly more general example which illustrates a connection between omega magmas and labelled, rooted binary trees. Let G be a directed graph and let X denote the 1-skeletal omega magma freely generated by the graph G . The set X_0 is just the set of vertices of G . The set X_1 is a set of ordered pairs. The first coordinate of such a pair is a finite sequence of composable edges of G which will generally

include some identity edges corresponding to elements of X_0 . The second component is a specific choice of a way of inserting parentheses into the sequence that is the first coordinate so as to represent a sequence of binary compositions yielding a single cell. The operation \bigcirc_0^1 is again the obvious operation derived from concatenation of compatible sequences of edges.

There is another way to represent the elements of X_1 in this example. Every such element corresponds to a unique rooted, binary tree whose leaves are linearly ordered and labelled. The labels of the leaves are just the edges appearing in the sequence that is the first coordinate of the chosen element of X_1 . Distinct elements of X_1 are represented by distinct labelled trees. However not every labelled tree corresponds to an element in X_1 ; for this to be so the labels must represent a sequence of elements that are compatible for composition in the order dictated by the tree structure. In this representation the operation \bigcirc_0^1 places side by side the trees representing the elements being composed and then joins them by adding a new root.

We can now define the notion of a strict omega category. These will be seen to be omega magmas whose partial operations are well behaved in that they satisfy the associative, interchange and identity laws.

Definition 5 *A **strict omega category** is an omega magma X satisfying the following axioms:*

1. **Associativity.** $(a \bigcirc_i^j b) \bigcirc_i^j c = a \bigcirc_i^j (b \bigcirc_i^j c)$

2. **Interchange.** *If $i < j < k$ then*

$$(a \bigcirc_j^k b) \bigcirc_i^k (c \bigcirc_j^k d) = (a \bigcirc_i^k c) \bigcirc_j^k (b \bigcirc_i^k d)$$

3. **Identity.** $a = id_j^i(dom_i^j a) \bigcirc_i^j a = a \bigcirc_i^j id_j^i(cod_i^j a)$

4. **Identity Interchange.** *If $i < j$ then*

$$id_{j+1}^j(a) \bigcirc_i^{j+1} id_{j+1}^j(b) = id_{j+1}^j(a \bigcirc_i^j b)$$

There is a category, *Strict_Category*, in which an object is a small, strict omega category and a morphism is a morphism of underlying omega magmas. This is clearly a full subcategory of *Omega_Magma*.

Any strict 2-category in the usual sense is in an obvious way a 2-skeletal, strict omega category. The standard example of such a strict 2-category is the 2-category in which objects are small categories, morphisms are functors and 2-cells natural transformations between functors.

Given a strict omega category X and a triple of indices $i < j < k$ one sees immediately that the sets X_i, X_j and X_k are the zero, one and two-cells respectively of a strict 2-category. In particular, for any pair of indices $i < j$ the sets X_i and X_j are the objects and morphisms respectively of an ordinary category in which the composition law is \circ_i^j .

1.2 Locally finitely presentable categories and essentially algebraic theories

In the Introduction to this paper we asserted that the reader needed only a familiarity with the basic language of categories and functors to understand our definition of weak omega category. Clearly the topics of this subsection require more expertise. However, the reader who feels that locally presentable categories and essentially algebraic theories are more technical baggage than he or she wishes to carry may safely skip this subsection and move on to Section 2. The material we shall now discuss is used only to construct certain adjunctions between categories. Nothing is lost if the reader is willing to take the existence of these adjunctions on faith.

Our basic references for locally presentable categories are [1] and [8]. For a treatment of essentially algebraic theories the reader may consult [1].

Definition 6 *An object a of a category A is **finitely presentable** if the functor $A(a, -)$ preserves directed colimits.*

The reader will recall that a directed colimit is a colimit over a diagram that is a directed poset. A poset is directed if each pair of elements has an upper bound.

Perhaps the most familiar example of a finitely presentable object in a category is a finitely presentable group (finite number of generators and relations) in the category of groups.

Definition 7 *A category A is **locally finitely presentable** if it is cocomplete and if there is a set K of finitely presentable objects such that each object a of A is a directed colimit of objects in K .*

Thus the category of groups is locally finitely presentable with the set K consisting of one representative from each isomorphism class of finitely presented groups.

There is a simple criterion for a functor between locally finitely presentable categories to be a right adjoint and this is the reason for our interest in such categories.

Theorem 8 ([1] theorem 1.66) *A functor between locally finitely presentable categories is a right adjoint if and only if it preserves limits and directed colimits.*

Occasionally we shall have reason to consider subcategories of locally finitely presentable categories and shall wish to prove that they are also locally finitely presentable. The following result will be useful in this regard.

Theorem 9 ([1] theorem 1.20) *A category is locally finitely presentable if and only if it is cocomplete and has a strong generator consisting of finitely presentable objects.*

We recall that a *strong generator* for a category A is a set G of objects with the following property. If b and c are objects of A and b is a proper subobject c then there is an object $a \in G$ and a morphism $f : a \rightarrow c$ that does not factor through b .

The preceding result asserts that every locally finitely presentable category is cocomplete, *but in fact such a category is also complete* ([1] theorem 1.28).

One way to show that a category is locally finitely presentable is to exhibit it as the category of models of an essentially algebraic theory in which all operations have finite arities, i.e. each depends only on a finite number of arguments. All of the categories defined earlier in this section can be seen to be locally finitely presentable for this reason. We next informally explain the concept of an essentially algebraic theory (see [1] chapter 3.D for the full story).

One defines an *essentially algebraic theory* by starting with a set of *sorts*. For example, the theory of globular sets is essentially algebraic and its i^{th} sort is the sort of i -cells. Thus this theory has one sort for each non-negative integer. The theory of omega magmas is also essentially algebraic and it has the same set of sorts as the theory of globular sets.

The second ingredient of an essentially algebraic theory is a set of *total operations*. The total operations of the theory of globular sets are the operations dom_i^j and cod_i^j for all pairs of integers $i < j$. The total operations of the theory of omega magmas are these plus the operations id_j^i for all pairs of integers $i < j$.

Essentially algebraic theories are distinguished from algebraic theories because they allow some operations to be only partially defined. All operations in the theory of globular sets are total and so this essentially algebraic theory is actually an algebraic theory. On the other hand the theory of omega magmas has *partial operations* \bigcirc_i^j defined only for those ordered pairs of j -cells (a, b) for which $cod_i^j a = dom_i^j b$. The important

point here is that the domain of definition for each partial operation is defined by equations involving only the total operations.

The final ingredient needed to define an essentially algebraic theory is a list of equations between operations (total and partial) which records the *axioms* of the theory.

A *model* of an essentially algebraic theory is determined by assigning to each of the theory's sorts a small set and to each total and partial operation an appropriate function between the (subsets of products of) sets assigned to sorts. These functions must satisfy the axioms of the theory. A *morphism between models* of the theory is simply a collections of functions, one for each sort, that maps the set corresponding to a sort in one model to the set corresponding to the same sort in the other model. These maps must of course commute with all the total and partial operations.

There is thus defined a category in which an object is a model of the essentially algebraic theory and a morphism is a morphism between models as described above. This category is called the *category of models of the theory*.

The important property of the category of models of an essentially algebraic theory is described by the following result.

Theorem 10 ([1] theorem 3.36) *A category is locally finitely presentable if and only if it is equivalent to the category of models of an essentially algebraic theory in which all arities are finite, each partial operation has domain defined by a finite number of equations, each such equation involves a finite number of variables and each axiom of the theory involves only a finite number of variables.*

It should be obvious from the definitions that *Glob_Set*, *Omega_Graph*, *Omega_Magma* and *Strict_Category* are categories of models of essentially algebraic theories and are moreover locally finitely presentable by the preceding result.

1.3 Some useful adjunctions

The definitions in section 1.1 show that there is a sequence of categories and forgetful functors

$$\begin{array}{ccccccc}
 & U_S^G & & U_G^M & & U_M^C & \\
 \textit{Glob_Set} & \leftarrow & \textit{Omega_Graph} & \leftarrow & \textit{Omega_Magma} & \leftarrow & \textit{Strict_Category}
 \end{array}$$

Each of the categories in this diagram is the category of models of an essentially algebraic theory and by theorem 10 is a locally finitely presentable category. In particular each of these categories is complete

and cocomplete. The forgetful functors are easy to describe. U_M^C forgets the axioms of a strict omega category, U_G^M forgets the composition laws of an omega magma and U_S^G forgets the identity functions of an omega graph. It is equally easy to see that each of these functors creates both limits and directed colimits (see [15, p.109] for the definition of creating limits) and therefore preserves such limits and colimits. Consequently theorem 8 assures us that each of these forgetful functors has a left adjoint. Thus we obtain a diagram

$$\begin{array}{ccccc} U_S^G & & U_G^M & & U_M^C \\ Glob_Set & \rightleftarrows & Omega_Graph & \rightleftarrows & Omega_Magma & \rightleftarrows & Strict_Category \\ L_C^S & & L_M^G & & L_C^M \end{array}$$

These left adjoints have explicit descriptions. The composite functor $L_C^S = L_C^M \circ L_M^G \circ L_S^G$ is isomorphic to the free omega category functor constructed by Batanin [5]. The functor L_C^S freely adjoins to a globular set the required identity elements. The composite $L_C^G = L_C^M \circ L_M^G$ is isomorphic to the free omega category functor (generated by an omega graph) constructed by Penon [18]. The functor L_M^G is implicit in Penon's construction of his "stretching" omega magmas [18]. Finally, the functor L_M^G assigns to an omega magma the coequalizer obtained by imposing the relations which must hold in any strict omega category.

These left adjoints will prove useful in analyzing the properties of our definition of weak omega categories and for constructing examples.

2 Weak omega categories

In this section we offer our definition of weak omega categories. In Sections 3,4 and 5 we shall explain how various types of higher dimensional categories that have appeared in the literature are all instances of the kind of weak omega category defined here.

Our first task is to define the concept of a *bridge relation*. Such relations can be thought of as carriers for the coherence data that defines an omega category structure on an omega magma.

First we establish some notation. Let X be an globular set and $a, b \in X_i$. We say that a and b are *parallel* and write $a \parallel b$ if $dom_{i-1}^i a = dom_{i-1}^i b$ and $cod_{i-1}^i a = cod_{i-1}^i b$. By convention, $a \parallel b$ holds for any pair of element in X_0 .

Definition 11 *Let X be an omega graph. A **bridge relation** R on X is a sequence of ternary relations*

$$R_i \subseteq X_i \times X_i \times X_{i+1}$$

defined for $i \geq 0$ and having the following properties:

1. $(a, b, c) \in R_i \Rightarrow a \parallel b$ and $\text{dom}_i^{i+1}c = a$ and $\text{cod}_i^{i+1}c = b$
2. $(a, a, \text{id}_{i+1}^i a) \in R_i$ for all $i \geq 0$ and for all $a \in X_i$
3. $(a, b, c), (a, b, d) \in R_i \Rightarrow c = d$

If X is also an omega magma, then a bridge relation on X is just a bridge relation on its underlying omega graph $U_G^M X$. If X is an omega magma or an omega graph with a bridge relation R then we shall call the pair (X, R) a **bridge magma** or a **bridge graph**.

We note that every omega graph has a diagonal bridge relation denoted by the symbol R^Δ . Here R_i^Δ has as its elements all triples of the form $(a, a, \text{id}_{i+1}^i a)$.

The significance of this terminology is evident. If $(a, b, c) \in R_i$ then the $(i + 1)$ -cell c is a “bridge” from the i -cell a to the i -cell b . One can imagine the parallel cells a, b as being the two banks of a river. In the literature bridge-cells like c are often part of what is commonly called a “contraction” but we think our terminology is more descriptive.

Definition 12 *Let X and Y be omega magmas and R^X, R^Y be bridge relations on X and Y . A morphism of omega magmas $F : X \rightarrow Y$ is called a **bridge morphism** if $(a, b, c) \in R_i^X \Rightarrow (Fa, Fb, Fc) \in R_i^Y$. We shall denote such a bridge morphism by $F : (X, R^X) \rightarrow (Y, R^Y)$.*

We shall require one more concept before we can define weak omega categories. This idea is due to Penon and is the essential feature of what he calls a categorical “stretching” [18].

Definition 13 *A **categorical Penon morphism** is a bridge morphism between omega magmas*

$$F : (X, R^X) \rightarrow (Y, R^Y)$$

satisfying the following conditions :

1. Y is a strict omega category and $R^Y = R^\Delta$
2. $(a, b, c) \in R_i^X \Rightarrow Fa = Fb$ and $Fc = \text{id}_{i+1}^i(Fa) = \text{id}_{i+1}^i(Fb)$
3. $a, b \in X_i$ and $a \parallel b$ and $Fa = Fb \Rightarrow \exists c$ and $(a, b, c) \in R_i^X$

We shall often write $F : X \rightarrow Y$ but say that F is a categorical Penon morphism when the bridge relation R^X is understood.

We are now ready to define weak omega categories. The reader will find it helpful to glance at the following diagram while reading the definition:

$$\mathbf{X} : \begin{array}{ccc} & \lambda^X & \kappa^X \\ X_1 & \xleftarrow{\quad} X_2 & \rightarrow X_3 \\ & \rho^X & \end{array}$$

Definition 14 A *weak omega category* \mathbf{X} consists of the following seven elements (see the above diagram):

1. An omega magma X_1 called the **underlying magma** of \mathbf{X}
2. An omega magma X_2 called the **coherence magma** of \mathbf{X}
3. A morphism $\lambda^X : X_2 \rightarrow X_1$ of omega magmas called the **coherence morphism** of \mathbf{X}
4. A morphism $\rho^X : X_1 \rightarrow X_2$ of omega graphs which splits λ^X in the category of omega graphs, i.e.

$$U_G^M(\lambda^X) \circ \rho^X = 1_{U_G^M X_1}$$

5. A bridge relation R^X on X_2
6. A strict omega category X_3
7. A categorical Penon morphism $\kappa^X : (X_2, R^X) \rightarrow (X_3, R^\Delta)$

We next define an omega functor in the obvious way.

Definition 15 Let \mathbf{X} and \mathbf{Y} be weak omega categories. An **omega functor** $\mathbf{F} : \mathbf{X} \rightarrow \mathbf{Y}$ is a triple of omega magma morphisms $\mathbf{F} = (F_1, F_2, F_3)$ with the following properties:

1. $F_i : X_i \rightarrow Y_i$
2. $F_2 : (X_2, R^X) \rightarrow (Y_2, R^Y)$ is a morphism of bridge magmas
3. The following diagram commutes in the category of omega magmas:

$$\begin{array}{ccccc} & \lambda^X & & \kappa^X & \\ & X_1 \leftarrow X_2 & \rightarrow & X_3 & \\ F_1 \downarrow & | & F_2 \downarrow & | & F_3 \downarrow \\ & \lambda^Y & & \kappa^Y & \\ & Y_1 \leftarrow Y_2 & \rightarrow & Y_3 & \end{array}$$

4. $F_2 \circ \rho^X = \rho^Y \circ F_1$ in the category of omega graphs.

Theorem 16 *The category, $\Omega\text{-Cat}$, in which an object is a weak omega category and a morphism is an omega functor is the category of models of an essentially algebraic theory. By theorem 10 it is locally finitely presentable and hence complete and cocomplete.*

Proof:

We only sketch the argument since it is a simple exercise in applying the definition of an essentially algebraic theory [1].

The sorts of the theory are indexed by pairs of integers (i, j) with $i = 1, 2, 3$ and $j \geq 0$. The sort indexed by the pair (i, j) is the sort of j -cells of the omega magma X_i .

The total operations are the domain, codomain, and identity functions of each of the three omega magmas together with the functions defining the morphisms λ^X, ρ^X and κ^X .

The partial operations are the composition operations for each of the three omega magmas and also the relations R_i^X comprising the bridge relation R^X . The relation R_i^X is in fact a partial function $X_i \times X_i \rightarrow X_{i+1}$ because it is single valued by definition 11, number 3.

The axioms of the theory are all equations. These state that the X_i are omega magmas for $i = 1, 2, 3$, that λ^X and κ^X are morphisms of omega magmas, that X_3 is a strict omega category, that R^X is a bridge relation, that κ^X is a categorical Penon morphism, that ρ^X is a morphism of omega graphs and that ρ^X splits λ^X in the category of omega graphs.

It is easy to check that a morphism of models is exactly an omega functor between weak omega categories. ■

There is an obvious forgetful functor from $\Omega\text{-Cat}$ to $\Omega\text{-Graph}$ that sends a weak omega category \mathbf{X} to the omega graph underlying the omega magma X_1 . Both categories are locally finitely presentable and it is easy to see that this forgetful functor preserves limits and directed colimits. It follows from theorem 8 that it has a left adjoint. We have not been able to identify this left adjoint and suspect it has no simple description. In any case $\Omega\text{-Cat}$ is very far from being monadic over $\Omega\text{-Graph}$. This is a consequence of Beck's "precise tripleability theorem" [15]. The forgetful functor from $\Omega\text{-Cat}$ fails to create split coequalizers (in fact it creates no colimits whatsoever) and hence by Beck's theorem $\Omega\text{-Cat}$ is not monadic over $\Omega\text{-Graph}$. The problem arises because this forgetful functor forgets far too much structure.

It is now easy to define a weak n-category.

Definition 17 A weak omega category \mathbf{X} is a **weak n -category** if its underlying omega magma X_1 is n -skeletal. (Recall that this means that X_1 has only identity cells above dimension n .) **Weak- n Cat** is the full subcategory of *Omega-Cat* in which an object is a weak n -category.

The definition makes it obvious that *Weak- n Cat* is itself the category of models of an essentially algebraic theory and as such is locally finitely presentable, complete and cocomplete.

3 Strict omega categories are weak omega categories

We begin exploring the properties of *Omega-Cat* by asking whether strict omega categories as defined in definition 5 are weak omega categories in the sense of definition 14.

The reader has probably already noticed that this is so, indeed in a trivial way. Let *Diag-Omega-Cat* denote the full subcategory of *Omega-Cat* with objects \mathbf{X} defined by the properties that $X_1 = X_2 = X_3$, that λ^X, ρ^X and κ^X are all the identity morphism and that $R^X = R^\Delta$. Note that for any such object the omega magmas X_i all are equal to the same strict omega category X_3 . Therefore *Diag-Omega-Cat* is isomorphic to *Strict-Cat*, the category of strict omega categories.

The definition of weak omega categories is illuminated in a more interesting way by considering the full subcategory *Weak-1Cat* of *Omega-Cat*. Define

$$U_M^W : \text{Weak-1Cat} \rightarrow \text{Omega-Magma}$$

to be the functor that sends an object \mathbf{X} to the omega magma X_1 . Now regard *Cat*, the category of small categories, as embedded in *Omega-Magma* by a functor that sends a category C to the strict omega category \widehat{C} which is identical to C in dimensions 0 and 1 and which is 1-skeletal. Let $\widehat{\text{Cat}}$ denote this full subcategory of *Omega-Magma*. Finally, define

$$S : \widehat{\text{Cat}} \rightarrow \text{Omega-Cat}$$

to be the functor that sends an object of $\widehat{\text{Cat}}$ to the obvious object of *Diag-Omega-Cat*. This object is clearly also an object of *Weak-1Cat*.

Theorem 18 The functor U_M^W has its image in $\widehat{\text{Cat}}$. In fact it is a retraction split by S .

Proof:

It will suffice to prove the first statement since the definition of S will then make the second trivial.

We must show that if \mathbf{X} is a weak omega category such that X_1 is 1-skeletal then the composition law \circ_0^1 and the identity function id_1^0 for X_1 define a structure of an ordinary category with the 0-cells of X_1 as objects and the 1-cells of X_1 as morphisms. We first prove that \circ_0^1 is an associative composition law.

Let f, g, h be 1-cells of X_1 such that $f \circ_0^1 g$ and $g \circ_0^1 h$ are defined. Since ρ^X is a morphism of omega graphs we conclude that $\rho^X f \circ_0^1 \rho^X g$ and $\rho^X g \circ_0^1 \rho^X h$ are defined in X_2 . Let $u \equiv (\rho^X f \circ_0^1 \rho^X g) \circ_0^1 \rho^X h$ and $v \equiv \rho^X f \circ_0^1 (\rho^X g \circ_0^1 \rho^X h)$. Clearly $u \parallel v$ and since X_3 is a strict omega category and κ^X is an omega magma morphism we conclude $\kappa^X u = \kappa^X v$. Since κ^X is a categorical Penon morphism it follows that $\exists c$ such that $(u, v, c) \in R_1^X$. Now $\lambda^X c$ must be an identity cell in X_1 because it is a 2-cell and X_1 has been assumed to be 1-skeletal. Consequently $\lambda^X u = \lambda^X v$ because c is a bridge-cell from u to v . Since λ^X is a morphism of omega magmas we conclude that

$$(\lambda^X \rho^X f \circ_0^1 \lambda^X \rho^X g) \circ_0^1 \lambda^X \rho^X h = \lambda^X \rho^X f \circ_0^1 (\lambda^X \rho^X g \circ_0^1 \lambda^X \rho^X h)$$

But λ^X is split by ρ^X . Thus $(f \circ_0^1 g) \circ_0^1 h = f \circ_0^1 (g \circ_0^1 h)$ as desired.

The proofs of the identity laws follow exactly the same pattern as the proof of the associative law. ■

Scholium. We wish to call to the reader's attention the pattern evident in the preceding proof. This is the standard method for proving that a desired coherence law must hold in the omega magma X_1 whenever \mathbf{X} is a weak omega category.

One first lifts the individual i -cells which are involved in the coherence law to X_2 using the omega graph morphism ρ^X . It is *very important* to note that this is a lifting of the *individual cells*, *not* of the composites they form in X_1 . One then reassembles the coherence diagram in X_2 using the lifts of these cells from X_1 . Next one observes that certain paths through this diagram define composite i -cells u and v in X_2 which are parallel. Applying κ^X one infers that the images of u, v are equalized by κ^X because X_3 is a strict omega category in which the desired coherence law holds as an equality. Since κ^X is a categorical Penon morphism one can then deduce that there is a unique $(i+1)$ -cell c that is a bridge from u to v in X_2 . The cell $\lambda^X c$ is then a bridge from $\lambda^X u$ to $\lambda^X v$ in X_1 . The facts that λ^X is an omega magma morphism split by the omega graph morphism ρ^X then yields the conclusion that the cell $\lambda^X c$ is a bridge between the desired composites in X_1 in the way illustrated by the concluding lines of the preceding proof.

In general the bridge-cell $\lambda^X c$ in X_1 will be neither an identity nor an isomorphism. However, it will be an equivalence in any reasonable sense as the following argument shows.

Since c is a bridge from u to v the axioms defining a categorical Penon morphism assure us that there is also a bridge c' from v to u . Then both $c \circ_i^{i+1} c'$ and $c' \circ_i^{i+1} c$ are defined and $c \circ_i^{i+1} c' \parallel id_{i+1}^i(u)$ while $c' \circ_i^{i+1} c \parallel id_{i+1}^i(v)$. One again appeals to the fact that κ^X is a categorical Penon morphism to deduce that κ^X equalizes both pairs of parallel cells. Consequently there is a unique $(i+2)$ -cell d that is the bridge from $c \circ_i^{i+1} c'$ to $id_{i+1}^i(u)$. Applying λ^X to these cells we find that $\lambda^X d$ is a bridge from $\lambda^X c \circ_i^{i+1} \lambda^X c'$ to $id_{i+1}^i(\lambda^X u)$. Thus, while the $(i+1)$ -cell $\lambda^X c$ is not an isomorphism there is a cell $\lambda^X c'$ such that the composite $\lambda^X c \circ_i^{i+1} \lambda^X c'$ and the identity cell $id_{i+1}^i(\lambda^X u)$ are bridged by an $(i+2)$ -cell $\lambda^X d$. Of course, the $(i+2)$ -cell $\lambda^X d$ is itself neither an identity nor an isomorphism. It does however have a “quasi-inverse” as did $\lambda^X c$. The composite of $\lambda^X d$ with its quasi-inverse can then be connected to $id_{i+2}^{i+1}(\lambda^X u)$ by yet another bridge cell of dimension $(i+3)$ and so forth. ■

4 Bicategories and weak 2-categories

For a definition of bicategories and a discussion of some of their properties the reader may consult [6,7,12,14,15,16]. Throughout this section we shall maintain as much consistency as possible with the bicategorical notation of [12].

Let *Bicat* denote the category in which an object is a bicategory and a morphism is strong homomorphism between bicategories (i.e. a morphism that preserves the operations and identities “on the nose”, not just up to isomorphism; in [12] this is called a strict homomorphism). Our aim in this section is to show that *Bicat* is a retract of *Weak_2_Cat*. Thus we shall construct a pair of functors

$$\begin{array}{ccc} & U_B^W & \\ & \uparrow & \\ Weak_2_Cat & \rightleftarrows & Bicat \\ & \downarrow & \\ & S_W^B & \end{array}$$

with the property that $U_B^W \circ S_W^B \simeq 1_{Bicat}$.

We first construct the functor U_B^W .

Theorem 19 *Let \mathbf{X} be a weak 2-category. Then its underlying omega magma, X_1 , can be structured as a bicategory for which the coherence data consists of 2-cells which are the images under λ^X of bridge-cells in X_2 .*

Proof:

In Section 7.1 we shall define for any weak omega category \mathbf{X} and any pair of $(i-1)$ -cells $a, b \in X_0$ a weak omega category $\mathbf{X}(a, b)$ in

which a 0-cell f is an i -cell of X_1 with domain a and codomain b . The composition laws of $\mathbf{X}(a, b)$ are the suitably reindexed restrictions of the composition laws of \mathbf{X} . However, when analyzing “hom categories” like $\mathbf{X}(a, b)$ it is usually convenient to retain the indexing of the composition laws of \mathbf{X} when discussing composition in $\mathbf{X}(a, b)$.

In the situation at hand, for any pair of 0-cells $a, b \in (X_1)_0$ we can consider the weak omega category $\mathbf{X}(a, b)$. Since \mathbf{X} is 2-skeletal we know that $\mathbf{X}(a, b)$ is 1-skeletal. Theorem 18 then asserts that $X(a, b)_1$ is in fact an ordinary category whose composition law is the restriction of the law \circ_1^2 in X_1 . Thus the first requirement X_1 must satisfy as a bicategory is met.

Our next task is to identify, for each triple of 0-cells $a, b, c \in (X_1)_0$ the composition functor

$$C_{abc} : X(a, b)_1 \times X(b, c)_1 \rightarrow X(a, c)_1$$

If (f, g) is an ordered pair of 0-cells (i.e. a pair of 1-cells of X_1) we define $C_{abc}(f, g) = f \circ_0^1 g$ where the operation is that of X_1 . Similarly, if (β, γ) is a pair of 1-cells (i.e. a pair of 2-cells of X_1) we define $C_{abc}(\alpha, \beta) = \alpha \circ_0^2 \beta$ where the operation is again that of X_1 .

We next show that C_{abc} defined in this way is in fact a functor. This is equivalent to showing that the interchange law holds as an equation in X_1 , i.e. that for a 4-tuple $(\beta, \gamma, \beta', \gamma')$ of compatible 2-cells in X_1 we have the equation (in X_1)

$$(\beta \circ_0^2 \gamma) \circ_1^2 (\beta' \circ_0^2 \gamma') = (\beta \circ_1^2 \beta') \circ_0^2 (\gamma \circ_1^2 \gamma')$$

It will suffice to exhibit a 3-cell of X_1 that is a bridge-cell from the left hand side of this equation to its right hand side. For by hypothesis, X_1 is 2-skeletal and hence any such bridge-cell must be an identity cell.

We produce such a bridge-cell by following the method described in the Scholium in Section 3. First use the omega graph morphism ρ^X to lift these four 2-cells from X_1 to X_2 . Then in X_2 assemble the two composites which correspond to those appearing in the last equation. Notice that these are parallel 2-cells in X_2 and that since κ^X is an omega magma morphism and X_3 is a strict omega category this pair of composite 2-cells is equalized by κ^X . It follows that this pair of composite cells is bridged by a 3-cell σ in X_2 . Next note that since ρ^X splits λ^X and since the latter is an omega magma morphism the images under λ^X of the two composite 2-cells in X_2 are precisely the 2-cells that appear on the right and left hand sides of the last equation. Consequently the 3-cell $\lambda^X \sigma$ is the desired bridge cell and must be an identity cell because X_1 is 2-skeletal.

Our next task is to produce the coherence data that makes X_1 a bicategory.

For any 4-tuple (a, b, c, d) of 0-cells of X_1 there are two obvious functors:

$$X(a, b)_1 \times X(b, c)_1 \times X(c, d)_1 \begin{array}{c} \xrightarrow{C_{abd} \circ (1 \times C_{bcd})} \\ \xrightarrow{C_{acd} \circ (C_{abc} \times 1)} \end{array} X(a, d)_1$$

We must construct a natural isomorphism

$$\alpha : C_{abd} \circ (1 \times C_{bcd}) \rightarrow C_{acd} \circ (C_{abc} \times 1)$$

called the associator of the bicategory. For any triple of 0-cells $(f, g, h) \in X(a, b)_1 \times X(b, c)_1 \times X(c, d)_1$ (these are 1-cells in X_1) we must define a 1-cell isomorphism (a 2-cell in X_1)

$$\alpha_{fgh} : f \circ_0^1 (g \circ_0^1 h) \rightarrow (f \circ_0^1 g) \circ_0^1 h$$

(where the operations are the operations of X_1). This is easily done using the technique of the Scholium of Section 3. One first uses this technique to exhibit a bridge-cell between these two composite 1-cells in X_1 . This bridge-cell must be an isomorphism because its composite with the bridge cell in the opposite direction is connected to an 2-cell identity by a three-cell. This three cell must itself be an identity cell because X_1 is 2-skeletal. Consequently the original bridge-cell is an isomorphism.

That these 2-cell isomorphisms are the components of a natural isomorphism of functors is also easy to prove using the method described in the Scholium. One simply lifts the 2-cells of X_1 comprising the diagrams which we must show are commutative to X_2 . After reassembling these diagrams in X_2 the by-now-standard argument shows that the two paths through this diagram define composite 2-cells in X_2 that are bridged by a 3-cell. Since X_1 is 2-skeletal the image of this 3-cell bridge under λ^X is an identity 3-cell in X_1 and so the diagram must commute in X_1 .

Precisely the same arguments produce natural isomorphisms

$$l_{ab} : C_{aab} \circ (I_a \times 1) \rightarrow 1_{X(a, b)_1}$$

$$r_{ab} : C_{abb} \circ (1 \times I_a) \rightarrow 1_{X(a, b)_1}$$

In these last two equations the functors $I_a \times 1$ and $1 \times I_a$ are respectively left and right composition of 1-cells in X_1 (0-cells in $X(a, b)_1$) with the identity 1-cell $id_1^0(a)$. ■

It is easy to see that when \mathbf{X} and \mathbf{Y} are weak 2-categories an omega functor $\mathbf{F} : \mathbf{X} \rightarrow \mathbf{Y}$ induces a strong homomorphism of the bicategory

structures we have just constructed on X_1 and Y_1 . This follows from the fact that the omega magma morphism F_2 is required to be a morphism of bridge magmas.

This completes the construction of the functor U_B^W .

We next construct the functor S_W^B from *Bicat* to *Weak_2_Cat*.

We begin by noting that any bicategory can be regarded as a 2-skeletal omega magma paired with associated coherence data. Forgetting this coherence data as well as the composition laws then defines a forgetful functor U_G^B from *Bicat* to *Omega_Graph*. Now *Bicat* is clearly the category of models of an essentially algebraic theory and it is easy to see that this forgetful functor creates limits and directed colimits and hence preserves them. Then theorem 8 tells us that U_G^B has a left adjoint L_B^G .

Now let Y be a bicategory regarded as a 2-skeletal omega magma provided with coherence data. We shall define a diagram of omega magmas, $S_W^B Y \equiv \mathbf{X}$, and show that this diagram is in fact a weak 2-category.

Let $X_1 \equiv Y$. Let $X_2 \equiv L_B^G \circ U_G^B(Y)$, the bicategory freely generated by the omega graph underlying Y . Define $\rho^X \equiv \eta U_G^B$, the component of the unit of the $L_B^G \dashv U_G^B$ adjunction corresponding to Y . Define $\lambda^X \equiv \varepsilon_Y$, the component of the counit of this adjunction corresponding to Y . Then ρ^X splits λ^X in the category of omega graphs by the triangle identities for the adjunction.

Next define $X_3 \equiv L_S^G \circ U_G^B(Y)$, the strict omega category freely generated by the omega graph $U_G^B Y$. Clearly this strict omega category is 2-skeletal and is therefore a freely generated, strict 2-category. Define $\kappa^X \equiv \eta U_M^S$, the component of the unit for the $L_S^M \dashv U_M^S$ adjunction between *Strict_Cat* and *Omega_Magma* that corresponds to the omega magma underlying $X_2 \equiv L_B^G \circ U_G^B(Y)$. In less mysterious terms κ^X is an omega magma morphism from the free bicategory to the free strict 2-category generated by the same omega graph. This morphism is induced by the omega magma congruence generated by the relations that must hold in every strict 2-category.

It should be clear that X depends functorially on the bicategory Y . Thus we can complete the construction of S_C^B by proving:

Theorem 20 *\mathbf{X} is a weak 2-category and the coherence data for Y is constructed from the images of the bridge-cells of X_2 under λ^X .*

Proof:

We shall first construct a bridge relation R^X on X_2 , the bicategory freely generated by the (2-skeletal) omega graph underlying Y . This construction is made possible by the coherence theorem for bicategories

[12,14,16]. It will be evident from the construction that the 2-cell coherence isomorphisms for Y are the images under λ^X of bridge cells defined by R^X .

For $i \neq 1$ define $R_i^X \equiv \{(a, a, id_{i+1}^i a) | a \in (X_2)_i\}$.

For $i = 1$ define

$$R_1^X \equiv \{(a, b, c) | a, b \in (X_2)_1, c \in (X_2)_2, c \text{ an isomorphism from } a \text{ to } b\}$$

The fact that X_2 is freely generated coupled with the coherence theorem for bicategories guarantees that

$$(a, b, c), (a, b, d) \in R_1^X \Rightarrow c = d$$

because the only 2-cell isomorphisms in $(X_2)_2$ are composites of coherence isomorphisms. (The point here is that freeness guarantees that the only automorphism of a 1-cell is the identity automorphism.) Moreover, such isomorphisms must necessarily connect parallel 1-cells. It follows that R^X is a bridge relation. Since λ^X is defined as a counit in *Bicat* it is a strong homomorphism of bicategories and hence necessarily sends the bridge elements determined by R^X to coherence isomorphisms of $X_1 = Y$.

It remains to prove that κ^X is a categorical Penon morphism for this bridge relation. If $(a, b, c) \in R_1^X$ then a, b are obviously equalized by κ^X . Moreover, the fact that X_3 is freely generated means that there are no relations among its 2-cells so that κ^X must send any 2-cell isomorphism to an identity cell. On the other hand the coherence theorem for bicategories tells us that if $a \parallel b$ are 1-cells of X_2 which are equalized by κ^X then there is a unique 2-cell isomorphism in X_2 from a to b . The fact that X_2 is freely generated means that if c is this isomorphism then $(a, b, c) \in \kappa^X$.

To verify the Penon condition for R_i^X , $i \neq 1$ we observe that by construction κ^X is an isomorphism on i -cells for $i = 0$ and for $i \geq 3$. We can thus complete the proof by showing that κ^X is faithful when restricted to 2-cells, i.e. that it does not equalize any parallel pair of 2-cells in X_2 . This fact will follow from the following easy lemma.

We first recall some terminology. A *clique* is an ordinary category equivalent to the terminal category; in other words, every pair of objects in a clique is connected by a unique isomorphism.

Lemma 21 *Let C be an ordinary category for which $ISO(C)$, the maximal subgroupoid of C whose morphisms are all isomorphisms, is a co-product (disjoint union) of cliques. Let \tilde{C} denote the category obtained from C by identifying isomorphic objects. Then the projection functor $C \rightarrow \tilde{C}$ is fully faithful.*

Proof of Lemma:

The hypothesis ensures that \tilde{C} is isomorphic to the category whose objects are the isomorphism classes $[a]$ of C and in which $\tilde{C}([a], [b]) \equiv C(a, b)$, the latter sets being canonically isomorphic for any choice of representatives by the clique hypothesis. ■

We apply the preceding lemma to the category $X_2(a, b)$, a, b 0-cells of X_2 . The fact that X_2 is freely generated as a bicategory coupled with the coherence isomorphism for bicategories ensures that the hypothesis of the lemma is satisfied. We conclude that κ^X is faithful when restricted to 2-cells thus completing the proof of the theorem. ■

5 The weak omega categories of Penon and Batanin

In this section we shall show that the weak omega categories defined by Penon in [18] are instances of our weak omega categories. We shall also offer an informal argument to support our claim that this is also true for the weak omega categories defined by Batanin [5]. In our opinion only our own lack of familiarity with the technical details of Batanin's work prevents us from making this argument rigorous.

5.1 The category *Prolixe*

Let us begin by considering the category *Pen_Mor* in which an object is a categorical Penon morphism (definition 13) and in which a morphism (H_W, H_Z) is a commutative square:

$$\begin{array}{ccc} & F & \\ & (X, R^X) \rightarrow (Y, R^\Delta) & \\ H_W & \downarrow & \downarrow H_Z \\ & (W, R^W) \rightarrow (Z, R^\Delta) & \\ & G & \end{array}$$

It is easy to see that *Pen_Mor* is the category of models of an essentially algebraic theory and as such is locally finitely presentable, complete and cocomplete.

There is a forgetful functor $U_G^P : \text{Pen_Mor} \rightarrow \text{Omega_Graph}$ which sends an Penon morphism (an object) to the omega graph which underlies the omega magma of it domain. This functor clearly preserves limits and directed colimits (in fact all filtered colimits) and so has a left adjoint L_P^G . Let $\mathbb{T} \equiv U_G^P \circ L_P^G$ denote the resulting monad on *Omega_Graph*. Penon defines a **prolix** to be a \mathbb{T} algebra and a morphism of prolixes to be a \mathbb{T} algebra morphism. There is therefore a category *Prolixe* and this is the category of Penon's weak omega categories. Since \mathbb{T} preserves filtered colimits *Prolixe* is cocomplete. However, it is not complete since it does

not contain Cartesian products. Thus *Prolixe* cannot be the category of models of any essentially algebraic theory.

Let us examine the left adjoint L_P^G in a little more detail. If Y is an omega graph $L_P^G Y$ is a morphism of bridge magmas

$$L_P^G Y : \text{domain } L_P^G Y \rightarrow \text{codomain } L_P^G Y$$

Using the universal property of the left adjoint one sees that codomain $L_P^G Y$ is just the strict omega category freely generated by the omega graph Y .

Here is an informal construction of the bridge magma which is the domain of the categorical Penon morphism $L_P^G Y$. (See part 2 of [18] for a formal construction.). Let $M(Y)_0$ denote the omega magma freely generated by Y and let $C(Y)$ denote the strict omega category freely generated by Y . Then $\eta_0 : M(Y)_0 \rightarrow C(Y)$ denotes the canonical morphism (unit of the omega magma-strict omega category adjunction). Begin the construction by formally adjoining a single 2-cell to $M(Y)_0$ for each ordered pair of parallel one cells equalized by η_0 . Extend η_0 over the new 2-cells by mapping them to the appropriate identities in $C(Y)$. Define R_1 to consist of those triples whose first two coordinates is an ordered pair of equalized, parallel 1-cells and whose third coordinate is the corresponding new 2-cell. Next “magnify” the resulting omega graph by freely adjoining all compositions with the new 2-cells and appropriate identity cells and extend η_0 in the obvious way over this new omega magma which we shall call $M(Y)_1$ (note that certain relations must also be added that say that composites consisting only of previously existing cells remain unaltered). Note that $M(Y)_0 \subset M(Y)_1$ as omega magmas and that these two omega magmas have the same 0- and 1-cells. We thus have a morphism $\eta_1 : M(Y)_1 \rightarrow C(Y)$. We repeat this process and obtain a new diagram $\eta_2 : M(Y)_2 \rightarrow C(Y)$ in which $M(Y)_2$ has the same 0-, 1- and 2-cells as $M(Y)_1$, $M(Y)_1 \subset M(Y)_2$ and R_2 has been defined. Continuing in this way and letting $M_P(Y)$ denote the colimit of the obvious directed diagram we obtain domain $L_P^G Y \equiv M_P(Y)$ and the morphism $L_P^G Y \equiv \eta_P(Y) : M_P(Y) \rightarrow C(Y)$.

5.2 *Prolixe* is a retract

We shall now show that *Prolixe* is a retract of full subcategory of *Omega_Cat*.

Define *Pen_Omega_Cat* to be the full subcategory of *Omega_Cat* in which an object \mathbf{X} has the following properties:

1. Define $Y \equiv U_G^M X_1$ then $\kappa^X = \eta_P(Y) \equiv L_P^G Y$
2. ρ^X is the unit of the *Omega_Graph* - *Pen_Mor* adjunction.

There is a functor

$$U_{PRO}^{POC} : Pen_Omega_Cat \rightarrow Prolixe$$

that sends an object \mathbf{X} to the \mathbb{T} algebra $U_G^M \lambda^X : U_G^M M_P(Y) \rightarrow U_G^M X_1$. The definition of U_{PRO}^{POC} on morphisms is the obvious one.

In the opposite direction, define a functor

$$S_{POC}^{PRO} : Prolixe \rightarrow Pen_Omega_Cat$$

as follows. If $h : \mathbb{T}Y \rightarrow Y$ is a \mathbb{T} algebra then $S_{POC}^{PRO}(h)$ is the following weak omega category \mathbf{X} :

1. The unit of \mathbb{T} gives Y the structure of an omega magma because this unit necessarily splits h . Define $X_1 \equiv Y$
2. Define $X_2 \equiv M_P(Y)$ and ρ^X to be the unit of \mathbb{T} and $\lambda^X \equiv h$. Note that this makes sense because h is necessarily an omega magma morphism by the definition of the omega magma structure on Y .
3. Define $\kappa^X \equiv \eta_P(Y) : M_P(Y) \rightarrow C(Y)$.

Clearly $U_{PRO}^{POC} \circ S_{POC}^{PRO} = 1_{Prolixe}$ and this shows that *Prolixe* is a retract of *Pen_Omega_Cat*.

5.3 Remarks on Batanin's weak omega categories

It is our view that it should be possible to adapt the preceding construction to show that the category of Batanin's weak omega categories [5] is a retract of some suitable subcategory of *Omega_Cat*. Let *Batanin_Cat* denote the category of algebras for Batanin's initial, contractible, higher dimensional operad with a system of contractions. This category is equivalent to the category of algebras for the monad \mathbb{B} canonically associated with the operad. The unit of the monad gives each algebra the structure of an omega graph. The system of compositions gives each algebra the structure of an omega magma. Thus the algebra $h : \mathbb{B}Y \rightarrow Y$ should correspond as above to the coherence arrow λ^X of a weak omega category. Now the category of strict omega categories is the category of algebras for Batanin's *terminal* operad and for its associated monad \mathbb{S} . The fact that \mathbb{S} is defined by the terminal operad should give rise to a canonical morphism of omega magmas $\mathbb{B}Y \rightarrow \mathbb{S}Y$. The omega magma $\mathbb{S}Y$ is a strict omega category and this last morphism should be a categorical Penon morphism because the operad defining \mathbb{B} is contractible. This construction should define a functor from *Batanin_Cat* to *Omega_Cat*. The functor in the opposite direction should be a simply defined version of the forgetful functor as it was in the case of *Prolixe*.

6 The stabilization conjecture

In [4] Baez and Dolan informally discuss a number of desirable properties a good theory of weak omega categories might have. Among these is a certain stability property which we now explain.

Definition 22 *A weak $(n + k)$ category \mathbf{X} is called **k -tuply monoidal** if each of the omega magmas X_1, X_2, X_3 has exactly one cell in each dimension $\leq k - 1$. We allow the possibility $n = \omega$. Let \mathbf{nCAT}_k denote the full subcategory of $\mathbf{Omega_Cat}$ in which an object is a k -tuply monoidal weak $(n + k)$ category.*

Baez and Dolan suggest that in a good theory of weak omega categories one should be able to construct a weak omega category nBD_k whose 0-cells are k -tuply monoidal objects like the objects of our category $nCat_k$. They hope that there would then be an omega functor

$$S : nBD_k \rightarrow nBD_{k+1}$$

(the stabilization functor) that would be some sort of equivalence provided $k \geq 2$.

We prove a stronger version of this stabilization property here.

Theorem 23 *For all $n \geq 0$ and $k \geq 2$ there is a functor $S_{n,k} : nCAT_k \rightarrow nCAT_{k+1}$ that is an isomorphism of categories.*

Proof:

The theorem will follow immediately from a similar result in the category of omega magmas.

A k -tuply monoidal omega magma is an omega magma that has exactly one cell in each dimension $\leq k - 1$. Let M be a k -tuply monoidal omega magma and assume $k \geq 2$. Define $W(M)$ to be the omega graph with a unique 0-cell $*$ and all of its higher dimensional identities and whose other i -cells are the $(i - 1)$ -cells of M with the obvious reindexing of the domain, codomain and identity functions of M .

Give $W(M)$ the structure of an omega magma in the following way. If a, b are j -cells which are compatible for composition over i -cells in $W(M)$ then a, b are by definition $(j - 1)$ -cells of M which are compatible for composition over $(i - 1)$ -cells in M . Then define the partial composition \circ_i^j in $W(M)$ by setting $\circ_i^j \equiv \circ_{i-1}^{j-1}$ where the latter law is composition in M . This defines composition laws \circ_i^j in $W(M)$ for $i \geq 1$. Define $\circ_0^j \equiv \circ_1^j$ where the right hand side is the law already defined in $W(M)$. Finally, define $id_j^0(*) \circ_0^j a \equiv a$ for all j -cells a and similarly for right composition with this identity.

It is easy to see that $W(M)$ is an omega magma and is just M “shifted up one dimension”. This W construction has an obvious inverse if $k \geq 3$ defined by “forgetting the unique 0-cell $*$ and all of its identities”. Clearly both the W construction and its inverse are functorial.

The theorem now follows from the observation that the W construction can be applied term by term to any k -tuply monoidal weak $(n + k)$ category and to the morphisms defining the category and to the bridge relation. The fact that the resulting functor $S_{n,k}$ is an isomorphism follows from the existence of the functorial inverse to the W construction for $k \geq 3$ ■

7 Constructing weak omega categories

In this section we shall discuss some methods for constructing weak omega categories from other weak omega categories, from omega magmas with special properties and from omega graphs.

The most basic observation is that Omega_Cat is the category of models of an essentially algebraic theory and is locally finitely presentable. It is therefore complete and cocomplete with respect to ordinary conical limits and colimits.

7.1 The weak omega category $\mathbf{X}(a, b)$

Next we construct a weak omega category $\mathbf{X}(a, b)$ for any weak omega category \mathbf{X} and any pair of $(i - 1)$ -cells a, b of X_1 . The 0-cells of the underlying omega magma $X(a, b)_1$ of $\mathbf{X}(a, b)$ will be the i -cells of X_1 with domain a and codomain b .

1. Define the omega magma $X(a, b)_1$ by defining the j -cells of $X(a, b)_1$ to be the $(j + i)$ -cells of X_1 which are mapped by dom_{i-1}^{j+i} to a and by cod_{i-1}^{j+i} to b . The domain, codomain and identity functions of $X(a, b)_1$ are just those of X_1 restricted and reindexed in the obvious way. The partial composition operations of $X(a, b)_1$ are again those of X_1 restricted and reindexed in the obvious way.
2. Define the omega magma $X(a, b)_2$ by defining the j -cells of $X(a, b)_2$ to be the $(j + i)$ -cells of X_2 which are mapped by $\text{dom}_{i-1}^{j+i} \circ \lambda^X$ to a and by $\text{cod}_{i-1}^{j+i} \circ \lambda^X$ to b . The remaining structure on $X(a, b)_2$ is defined in just the same way as it was for $X(a, b)_1$.
3. Clearly the omega magma morphism $\lambda^{X(a,b)}$ and the omega graph morphism $\rho^{X(a,b)}$ can be defined as the restrictions of the morphisms λ^X and ρ^X and will have the required properties.

4. Define the strict omega category $X(a, b)_3$ to have as its j -cells *all* of the $(j + i)$ -cells of X_3 . The remaining structure can be defined as it was for $X(a, b)_1$
5. The bridge relation $R^{X(a, b)}$ on $X(a, b)_2$ is just the bridge relation R^X restricted and reindexed in the obvious way.
6. Finally, the omega magma morphism $\kappa^{X(a, b)}$ is just the obvious restriction of κ^X and is clearly a categorical Penon morphism.

This construction of the weak omega category $\mathbf{X}(a, b)$ is functorial in the following sense. Let $\mathbf{F} : \mathbf{X} \rightarrow \mathbf{Y}$ be an omega functor and a, b a pair of $(i - 1)$ -cells of X_1 . Then \mathbf{F} “restricts” in the obvious way to an omega functor

$$\mathbf{F}(a, b) : \mathbf{X}(a, b) \rightarrow \mathbf{Y}(F_1 a, F_1 b)$$

7.2 Weak omega categories from omega graphs

7.2.1 The functor P

It can be useful to have a method for associating in a functorial way weak omega categories with omega graphs and with certain diagrams of omega graphs.

In Section 5 we constructed a functor L_P^G , left adjoint to a forgetful functor, from the category of omega graphs to the category of categorical Penon morphisms. If Y is an omega graph recall that we denoted $L_P^G Y$ by

$$L_P^G Y \equiv \eta_P(Y) : M_P(Y) \rightarrow C(Y)$$

We shall now show how this functor can be used to construct a functor from omega graphs to weak omega categories.

We shall define a functor

$$P : \text{Omega_Graph} \rightarrow \text{Omega_Cat}$$

and call the weak omega category $P(Y)$ the Penon category associated with Y . Temporarily denote $P(Y)$ by \mathbf{X} . Then define:

1. $X_1 = X_2 \equiv M_P(Y)$, $\lambda^X = \rho^X \equiv 1_{M_P(Y)}$
2. $X_3 \equiv C(y)$, $\kappa^X \equiv \eta_P(Y)$

It should be obvious that this definition extends to morphisms without difficulty. As far as we have been able to determine P is not left adjoint to any functor resembling a forgetful functor defined on all of Omega_Cat .

7.2.2 The functor P_A^\rightarrow

There is a variant of the Penon category construction which we shall find useful in Section 9. Recall the category of categorical Penon morphisms, Pen_Mor from the beginning of Section 5. An object is a categorical Penon morphism (definition 13) and a morphism is the obvious commutative square. Fix a strict omega category A and consider the subcategory $Pen_Mor(A)$ in which every object has codomain A and in which every morphism is the identity functor of A on its codomain leg. Consider the subcategory of $Omega_Graph^\rightarrow$, the arrow category of $Omega_Graph$, in which every object has codomain $U_G^C A$, the omega graph underlying A and in which every morphism has the identity of $U_G^C A$ as its codomain leg. Denote this subcategory by $Omega_Graph^{\rightarrow UA}$. There is an obvious forgetful functor

$$U_{GMA}^{PMA} : Pen_Mor(A) \rightarrow Omega_Graph^{\rightarrow UA}$$

We shall argue that this functor has a left adjoint. From an informal point of view this is obvious from the heuristic construction of the omega magma morphism $\eta_P : M(Y)_P \rightarrow C(Y)$ in Section 5. This construction started with the canonical arrow η_0 from the omega magma freely generated by the omega graph Y to the strict omega category generated by Y . But the construction would clearly make sense starting with any arrow from this free omega magma to any strict omega category; this is the data provided by objects of the category $Omega_Graph^{\rightarrow UA}$.

A formal argument for the existence of the left adjoint runs as follows. The forgetful functor obviously creates limits and directed colimits and hence preserves them. (One should note that the categorical product in each category is in fact the pullback over the codomain.) To apply theorem 8 we would like to show that both categories in question are locally finitely presentable. This can be done in two different ways.

One can show that each of these categories is the category of models of an essentially algebraic theory. This is rather tedious because one must index sorts in an unusual way. Sorts are types of cells in the domain magma or graph. Such a sort is indexed by its dimension and also by its image cell in the codomain. This leads to complicated bookkeeping when it comes to defining total and partial operations.

A more straightforward approach is to apply theorem 9 which characterizes locally finitely presentable categories. It is easy to show that each of these categories is cocomplete. If we can show that each has a strong generator then it will follow that each is locally finitely presentable. The idea now is to use the strong generators of the ambient categories (which exist because the ambient categories are locally finitely presentable) to

construct the desired strong generators. We illustrate this construction for $Pen_Mor(A)$.

Let G be strong generator of Pen_Mor and $Y_1 \rightarrow Y_2$ one of its elements (which must be a finitely presentable object). For each possible strict omega functor between strict omega categories $Y_2 \rightarrow A$ we consider the composite morphism of omega magmas $Y_1 \rightarrow A$. This will be a finitely presentable object in $Pen_Mor(A)$ if $Y_1 \rightarrow Y_2$ is finitely presentable in Pen_Mor . Thus we have a set of finitely presentable objects of $Pen_Mor(A)$ whose elements are indexed by strict omega functors $Y_2 \rightarrow A$. There is one such set for each element of the generator G of Pen_Mor . Taking the union of all these sets over the elements of G gives us a set of objects that we hope will be a strong generator for $Pen_Mor(A)$. That it is indeed a strong generator follows immediately from the fact that G is a strong generator of Pen_Mor .

Thus we can conclude that the forgetful functor

$$U_{GMA}^{PMA} : Pen_Mor(A) \rightarrow Omega_Graph^{\rightarrow UA}$$

has a left adjoint L_{PMA}^{GMA} . Now let $Omega_Cat(A)$ denote the full subcategory of $Omega_Cat$ in which an object \mathbf{X} has the property that $X_3 = A$. We define a functor

$$P_A^{\rightarrow} : Omega_Graph^{\rightarrow UA} \rightarrow Omega_Cat(A)$$

as follows. Let $f : Y \rightarrow UA$ denote an object of $Omega_Graph^{\rightarrow UA}$ and denote by \mathbf{X} the object $P_A^{\rightarrow} f$.

1. Set $X_1 = X_2 \equiv domain\ L_{PMA}^{GMA}(f)$, $\lambda^X = \rho^X \equiv 1_{domain\ L_{PMA}^{GMA}(f)}$
2. Set $\kappa^X \equiv L_{PMA}^{GMA}(f)$

This construction is clearly functorial in f and defines the functor P_A^{\rightarrow} .

7.3 Categorical equivalence relations

In applications of this theory (see part II of this work which will be forthcoming) it is often convenient to have a criterion which identifies a given omega magma M as the domain of a (yet-to-be-constructed) categorical Penon morphism f . Any such omega magma then determines a weak omega category by setting $X_1 = X_2 = M$, $\lambda^X = \rho^X = 1_M$ and $\kappa^X = f$.

Definition 24 *An omega magma equivalence relation on M is an omega submagma $E \subseteq M \times M$ such that E_j is an equivalence relation on the j -cells M_j of M for all $j \geq 0$.*

Recall that $U_G^M M$ is the underlying omega graph of M while $L_M^G \circ U_G^M(M)$ is the omega magma freely generated by this omega graph and $L_G^C \circ U_G^M(M)$ is the strict omega category freely generated by this graph. Let

$$\varepsilon_M : L_M^G \circ U_G^M(M) \rightarrow M$$

denote the counit of the adjunction and

$$\eta_M : L_M^G \circ U_G^M(M) \rightarrow L_G^C \circ U_G^M(M)$$

denote the unit of the $L_G^M \dashv U_M^C$ adjunction.

Definition 25 *An omega magma equivalence relation E on M is **categorical** if for all elements $u, v \in L_M^G \circ U_G^M(M)$ the following implication holds: $\eta_M u = \eta_M v \implies (\varepsilon_M u, \varepsilon_M v) \in E$.*

In less formal terms, we say that an omega magma equivalence relation is categorical if any pair of elements which are equalized by *every* morphism to a strict omega category are E equivalent in M .

Definition 26 *An omega magma equivalence relation E on M is **sharp** if $a, b \in M$ and $(\text{cod}_k^j a, \text{dom}_k^j b) \in E \implies \exists a', b' \in M$ and $(a, a'), (b, b') \in E$ and $\text{cod}_k^j a' = \text{dom}_k^j b'$.*

Now if E is any omega magma equivalence relation on M we can form the coequalizer $M//E$ of the two projections. This coequalizer exists because the category of omega magmas is cocomplete. If E is sharp then this coequalizer will not contain “extra arrows” which in general will arise when taking the quotient by an arbitrary equivalence relation.

Theorem 27 *Let E be a sharp, categorical, omega magma equivalence relation on M . Then $(M//E)_j \simeq M_j/E_j$ and $M//E$ is a strict omega category.*

Proof:

It will suffice to show that the coequalizer of the two projections from $U_G^M E$ in the category of omega graphs can be given the structure of a strict omega category in such a way that the quotient map from $U_G^M M$ to the coequalizer is in fact an omega magma morphism. If this can be done then this strict omega category is easily seen to have the universal property of the desired coequalizer and hence must be isomorphic to it.

Now the coequalizer $U_G^M E \rightrightarrows U_G^M M$ is just the omega graph $U_G^M M//U_E^M E$ whose j -cells are the equivalence classes of M_j by the equivalence relation E_j ; this set of equivalence classes was denoted in the

statement of the theorem by M_j/E_j . We now show that $U_G^M M // U_E^M E$ has an omega magma structure for which the quotient map is an omega magma morphism.

Let $[a], [b]$ be equivalence classes in M_j/E_j such that $\text{cod}_k^j[a] = \text{dom}_k^j[b]$. Since $U_G^M M // U_E^M E$ is an omega graph we see that $(\text{cod}_k^j a, \text{dom}_k^j b) \in E$. Since E is sharp there exists $a' \in [a], b' \in [b]$ such that $\text{cod}_k^j a' = \text{dom}_k^j b'$. Then define $[a] \circ_k^j [b] \equiv [a' \circ_k^j b']$. Because E is an omega submagma of M this definition is independent of the choices made. Clearly the quotient map $U_G^M M \rightarrow U_G^M M // U_E^M E$ is the underlying omega graph morphism of a morphism of omega magmas.

The fact that the omega magma structure defined on $U_G^M M // U_E^M E$ gives it the structure of a strict omega category follows immediately from the fact that the relation E was assumed to be categorical. ■

Now let us assume that the omega magma M comes equipped with a bridge relation R^M and a sharp, categorical, omega magma equivalence relation E .

Definition 28 R^M is a *witness* to E if

1. $(a, b, c) \in R_j^M \implies (a, b) \in E$ and $(c, \text{id}_{j+1}^j a), (c, \text{id}_{j+1}^j b) \in E$
2. $a \parallel b$ and $(a, b) \in E_j \implies \exists c$ and $(a, b, c) \in R_j^M$

The following result is a trivial consequence of the last definition and of the preceding theorem.

Theorem 29 Let (M, R^M) be a bridge magma, let E be a sharp, categorical, omega magma equivalence relation on M and assume R^M is a witness to E . Then the quotient map induces a categorical Penon morphism.

$$\kappa_E : (M, R^M) \rightarrow (M // E, R^\Delta)$$

8 Pseudo-functors

In this section we develop a theory of omega pseudo-functors between weak omega categories. Intuitively, an omega pseudo-functor differs from an omega functor in that the former preserves operations and identities only up to equivalence instead of “on the nose”. It is our view that omega pseudo-functors are the most natural type of morphism between weak omega categories. We show that omega pseudo-functors are ubiquitous by showing that in any weak omega category \mathbf{X} composition with an i -cell defines a pair of omega pseudo-functors that are not in general omega functors. More specifically, we show that if a, b, c is a triple of

$(i - 1)$ -cells in X_1 and if h is an i -cell with $\text{dom}_{i-1}^i h = b$ and $\text{cod}_{i-1}^i h = c$ then there is an omega pseudo-functor

$$\Theta(\mathbf{h}) : \mathbf{X}(a, b) \rightarrow \mathbf{X}(a, c)$$

which is defined by right composition with h over the $(i - 1)$ -cell b . Of course the analogous result holds for left composition as well.

We also show that an omega pseudo-functor between 2-skeletal weak omega categories is a homomorphism in the standard sense (which we shall call “classical”). The classical homomorphisms which arise in this way from omega pseudo-functors have a special property of being “proper”. We know of no classical homomorphism that is not also proper but we expect that such examples can be constructed. In any case non-proper homomorphisms do not seem to arise naturally. We close this section by showing that any proper, classical pseudo-functor is an omega pseudo-functor.

The reader will notice that to establish the correspondence between omega pseudo-functors and classical, proper homomorphisms in the 2-skeletal case we have to employ some lengthy arguments. This suggests to us that our definition of pseudo-functor is indeed a non-trivial change of viewpoint even for homomorphisms between bicategories.

8.1 Definition of omega pseudo-functor

Let \mathbf{X}, \mathbf{Y} be weak omega categories and let $\mathbf{F} = (F_1, F_2, F_3)$ be a triple of morphisms of *globular sets*. Consider the following diagram of globular sets:

$$\begin{array}{ccccc} & \lambda^X & & \kappa^X & \\ & X_1 \leftarrow X_2 \rightarrow X_3 & & & \\ F_1 \downarrow & & F_2 \downarrow & & F_3 \downarrow \\ & \lambda^Y & & \kappa^Y & \\ Y_1 \leftarrow & & Y_2 \rightarrow & & Y_3 \end{array}$$

Definition 30 \mathbf{F} is an *omega pseudo-functor* if

1. F_3 is a morphism of omega graphs
2. the F_i make the above diagram commute in the category of globular sets
3. for all a, b such that $a \circ_{j-1}^j b$ is defined in X_2

$$\kappa^Y(F_2 a \circ_{j-1}^j F_2 b) = \kappa^Y F_2(a \circ_{j-1}^j b)$$

We note that we do **not** require the equality $F_2 \circ \rho^X = \rho^Y \circ F_1$ to hold for \mathbf{F} to be an omega pseudo-functor.

Proposition 31 *Let $\mathbf{F} : \mathbf{X} \rightarrow \mathbf{Y}$ and $\mathbf{G} : \mathbf{Y} \rightarrow \mathbf{Z}$ be omega pseudo-functors. Then $\mathbf{G} \circ \mathbf{F}$ is an omega pseudo-functor.*

Proof:

The composite $G_3 \circ F_3$ is a morphism of omega graphs so it remains only to verify condition 3 of the definition. To this end let $a, b \in X_2$ and suppose that $a \circ_{j-1}^j b$ is defined. Since \mathbf{F} is an omega pseudo-functor we know that

$$\kappa^Y(F_2a \circ_{j-1}^j F_2b) = \kappa^Y \circ F_2(a \circ_{j-1}^j b)$$

Since $G_3 \circ \kappa^Y = \kappa^Z \circ G_2$ we conclude

$$\kappa^Z \circ G_2(F_2a \circ_{j-1}^j F_2b) = \kappa^Z \circ G_2 \circ F_2(a \circ_{j-1}^j b).$$

On the other hand the fact that \mathbf{G} is an omega pseudo functor yields

$$\kappa^Z((G_2 \circ F_2a) \circ_{j-1}^j (G_2 \circ F_2b)) = \kappa^Z \circ G_2(F_2a \circ_{j-1}^j F_2b).$$

Therefore

$$\kappa^Z((G_2 \circ F_2a) \circ_{j-1}^j (G_2 \circ F_2b)) = \kappa^Z \circ G_2 \circ F_2(a \circ_{j-1}^j b)$$

as desired. ■

From the preceding proposition we conclude that there is a category PF_Omega_Cat in which an object is a weak omega category and a morphism is an omega pseudo-functor. PF_Omega_Cat contains $Omega_Cat$ as a subcategory. As a category PF_Omega_Cat is not nearly so well-behaved as $Omega_Cat$. For example, it is neither complete nor cocomplete. Thus it is not locally presentable and is not the category of models of an essentially algebraic theory.

We note one more obvious fact. Let $\mathbf{F} : \mathbf{X} \rightarrow \mathbf{Y}$ be an omega pseudo-functor. Let a, b be any pair of $(i-1)$ -cells of X_1 . Then the appropriate restriction of \mathbf{F} defines an omega pseudo-functor

$$\mathbf{F}(a, b) : \mathbf{X}(a, b) \rightarrow \mathbf{Y}(F_1a, F_1b)$$

8.2 Pseudo-functors defined by composition in a weak omega category

We next show that every i -cell $h, i \geq 1$, in the underlying omega magma of a weak omega category \mathbf{X} defines a pair of omega pseudo-functors

which correspond to left and right composition with h . We shall prove this for right composition since the case of left composition is proved in exactly the same way.

Let a, b, c be three $(i-1)$ -cells of X_1 and h an i -cell satisfying

$$\text{dom}_{i-1}^i h = b, \text{cod}_{i-1}^i h = c$$

Recall the construction of the weak omega categories $\mathbf{X}(a, b), \mathbf{X}(a, c)$ from Section 7. We shall define a triple of globular set morphisms $\Theta(h)_j$, $j = 1, 2, 3$ with

$$\Theta(h)_j : X(a, b)_j \rightarrow X(a, c)_j$$

and then we shall show that $\Theta(\mathbf{h}) \equiv (\Theta(h)_j)$ is an omega pseudo-functor. Let \circ denote composition in any of the three omega magmas defining \mathbf{X} .

1. Let $f \in (X(a, b)_1)_j$. By definition this means that $f \in (X_1)_{i+j}$. Define

$$\Theta(h)_1 f \equiv f \circ_{i-1}^{i+j} id_{i+j}^i h$$

2. Let $f \in (X(a, b)_2)_j$. Then $f \in (X_2)_{i+j}$. Define

$$\Theta(h)_2 f \equiv f \circ_{i-1}^{i+j} id_{i+j}^i (\rho^X h)$$

3. Let $f \in (X(a, b)_3)_j$. Then $f \in (X_3)_{i+j}$. Define

$$\Theta(h)_3 f \equiv f \circ_{i-1}^{i+j} id_{i+j}^i (\kappa^X \rho^X h)$$

We note that since X_3 is a strict omega category the morphism of globular sets $\Theta(h)_3$ is in fact a strict functor between strict omega categories because the interchange law and identity laws hold as equations in X_3 . We shall see that this is a special property of omega pseudo-functors of the form $\Theta(\mathbf{h})$ which is not shared by the general omega pseudo-functor.

Proposition 32 $\Theta(\mathbf{h}) : X(a, b) \rightarrow X(a, c)$ is an omega pseudo-functor.

Proof:

One easily sees that the triple of globular set morphisms $(\Theta(h)_1, \Theta(h)_2, \Theta(h)_3)$ makes the relevant diagram of globular sets commute. Since $\Theta(h)_3$ is a morphism of strict omega categories it is a fortiori a morphism of omega graphs. It thus only remains to check condition 3 in the definition of omega pseudo-functor.

Let $f, g \in (X(a, b)_2)_j$ and suppose that $f \circ_{j-1}^j g$ is defined. Then $f, g \in (X_2)_{i+j}$ and $f \circ_{i+j-1}^{i+j} g$ is defined and

$$\Theta(h)_2(f \circ_{j-1}^j g) \equiv (f \circ_{i+j-1}^{i+j} g) \circ_{i-1}^{i+j} id_{i+j}^i h$$

where the left hand side of this equation is an expression involving operations and elements of $X(a, b)_2$ while the right hand side is an expression involving operations and elements of X_2 . Define

$$u \equiv (f \circ_{i+j-1}^{i+j} g) \circ_{i-1}^{i+j} id_{i+j}^i h.$$

Note that

$$\Theta(h)_2(f) \circ_{j-1}^j \Theta(h)_2(g) = (f \circ_{i-1}^{i+j} id_{i+j}^i h) \circ_{i+j-1}^{i+j} (g \circ_{i-1}^{i+j} id_{i+j}^i h)$$

where again the left hand side denotes operations and elements in $X(a, b)_2$ while the right hand side denotes operations and elements in X_2 . Define

$$v \equiv (f \circ_{i-1}^{i+j} id_{i+j}^i h) \circ_{i+j-1}^{i+j} (g \circ_{i-1}^{i+j} id_{i+j}^i h).$$

We must show that $\kappa^X u = \kappa^X v$.

To this end we define an element u' of $(X_2)_{i+j}$

$$u' \equiv (f \circ_{i+j-1}^{i+j} g) \circ_{i-1}^{i+j} (id_{i+j}^i h \circ_{i+j-1}^{i+j} id_{i+j}^i h).$$

Note that since identity laws hold strictly in X_3 and since κ^X is an omega magma morphism $\kappa^X u = \kappa^X u'$. But the interchange law also holds as an equation in X_3 so $\kappa^X u' = \kappa^X v$. ■

8.3 Omega pseudo-functors and homomorphisms

We shall now examine the relationship between omega pseudo-functors and classical pseudo-functors (usually called homomorphism in the bicategory case and pseudo-functors in the strict 2-category case). Recall that the results of Section 4 show that the category of bicategories and strong homomorphisms is a retract of the category of 2-skeletal, weak omega categories and omega functors. We cannot prove an analogous result in which homomorphisms replace strong homomorphism and omega-pseudo functors replace omega functors. In fact we strongly suspect that such a retraction may not exist.

In any event what we will show is that the F_1 component of any omega pseudo functor between 2-skeletal, weak omega categories \mathbf{X} and \mathbf{Y} defines a classical homomorphism between the bicategories X_1 and Y_1 .

The bicategory homomorphisms which arise in this way have a special property we call “properness”. We have been unable to construct or to find any example of a homomorphism that is not proper. However, the definition of homomorphism does not apparently preclude the existence of such an example.

In the next subsection we shall also show that for any classical homomorphism h between bicategories A and B there exist 2-skeletal, weak omega categories \mathbf{X} and \mathbf{Y} and omega pseudo-functor \mathbf{F} between them such that $X_1 = A$, $Y_1 = B$ and $F_1 = h$. The proof of this last result is surprisingly lengthy and is evidence that our definition of omega pseudo-functor represents a significant change of viewpoint from the classical one.

Let A, B be classical bicategories. We refer to [7,12,14] for the definition of a homomorphism from A to B . Essentially a homomorphism differs from a strong homomorphism in that it preserves identities and operations only up to isomorphisms in B . These isomorphisms must fit together “coherently” and must thus satisfy some axioms. These axioms make working with composite homomorphisms a complicated task. Our definition of omega pseudo-functor offers a path through these complications.

Let $H : A \rightarrow B$ be a homomorphism between classical bicategories. Part of the definition of H is coherence data which must satisfy some axioms. This coherence data has two components:

1. For each pair of 1-cells $f, g \in A$ such that $f \circ_0^1 g$ is defined there is given a 2-cell isomorphism

$$\phi_{f,g} : Hf \circ_0^1 Hg \rightarrow H(f \circ_0^1 g)$$

2. For each 0-cell $a \in A$ there is given a 2-cell isomorphism

$$\phi_a : id_1^0(Ha) \rightarrow H(id_1^0 a).$$

The 2-cell isomorphisms $\phi_{f,g}$ and ϕ_a are in fact components of a pair of natural isomorphisms between functors. These natural transformations must satisfy some axioms which we shall not display here (but see [7,12]).

Definition 33 *A homomorphism $H : A \rightarrow B$ between classical bicategories is called **proper** if the associated coherence data has the following properties:*

1. *for all 1-cells f, g of A*

$$Hf \circ_0^1 Hg = H(f \circ_0^1 g) \implies \phi_{f,g} = id_2^1(H(f \circ_0^1 g))$$

2. for all 0-cells a of A

$$id_1^0(Ha) = H(id_1^0 a) \implies \phi_a = id_2^1(H(id_1^0 a))$$

3. for all 1-cells f, f', g, g' of A

$$f \circ_0^1 g = f' \circ_0^1 g' \text{ and } Hf \circ_0^1 Hg = Hf' \circ_0^1 Hg' \implies \phi_{f,g} = \phi_{f',g'}$$

4. for all 0-cells a, b of A

$$Ha = Hb \text{ and } H(id_1^0 a) = H(id_1^0 b) \implies \phi_a = \phi_b$$

Theorem 34 *Let $\mathbf{F} : \mathbf{X} \rightarrow \mathbf{Y}$ be an omega pseudo-functor between 2-skeletal, weak omega categories. Then there is coherence data $\phi_{f,g}$ and ϕ_a which exhibits F_1 as a homomorphism between the bicategories X_1 and Y_1 . This homomorphism is proper.*

Proof:

The fact that X_1 and Y_1 are bicategories was proven in Section 4. We shall construct the coherence data $\phi_{f,g}$ and ϕ_a and then outline the method the reader may follow to satisfy himself that all necessary diagrams involving this coherence data commute in Y_1 . It will be clear from the construction of this coherence data that F_1 must be a proper homomorphism.

Let f, g be 1-cells of X_1 such that $f \circ_0^1 g$ is defined. Since \mathbf{F} is an omega pseudo-functor we know that κ^Y equalizes the 1-cells u, v of Y_2 where these cells are defined by the equations

$$u \equiv (F_2 \circ \rho^X) f \circ_0^1 (F_2 \circ \rho^X) g$$

$$v \equiv (F_2 \circ \rho^X)(f \circ_0^1 g).$$

Since these two cells are also parallel we conclude that there exists a (unique) 2-cell $\phi'_{f,g} \in Y_2$ such that

$$(u, v, \phi'_{f,g}) \in R_1^Y.$$

(Later in the proof it will be helpful for the reader to recall that $\kappa^Y \phi'_{f,g}$ is an identity 2-cell of Y_3 .) Define $\phi_{f,g} \equiv \lambda^Y \phi'_{f,g}$. Since λ^Y is an omega magma morphism that is split by the omega graph morphism ρ^Y and since $\lambda^Y \circ F_2 = F_1 \circ \lambda^X$ as morphisms of globular sets we conclude that

$$dom_1^2 \phi_{f,g} = F_1 f \circ_0^1 F_1 g$$

and

$$\text{cod}_1^2 \phi_{f,g} = F_1(f \circ_0^1 g)$$

We define the coherence data ϕ_a in a similar way. Here the assumption that F_3 is a morphism not just of globular sets but of omega graphs (and hence preserves identities) will play a crucial role.

Let a denote a 0-cell of X_1 and consider the 1-cells of Y_2 defined by the equations

$$\begin{aligned} u &\equiv F_2 \text{id}_1^0(\rho^X a) \\ v &\equiv \text{id}_1^0(F_2 \rho^X a). \end{aligned}$$

These two 1-cells are obviously parallel. We claim that they are equalized by κ^Y . To see this first note that by hypothesis $\kappa^Y \circ F_2 = F_3 \circ \kappa^X$. The latter morphism is actually a morphism of omega graphs because F_3 is assumed to be a morphism of omega graphs. Thus $\kappa^Y \circ F_2$ is a morphism of omega graphs. It follows that

$$\kappa^Y u = \kappa^Y F_2(\text{id}_1^0(\rho^X a)) = \text{id}_1^0(\kappa^Y F_2 \rho^X a) = \kappa^Y \text{id}_1^0(F_2 \rho^X a) = \kappa^Y v.$$

Consequently we deduce the existence of a (unique) 2-cell ϕ'_a of Y_2 such that

$$(u, v, \phi'_a) \in R_1^Y.$$

We define $\phi_a \equiv \lambda^Y \phi'_a$ and observe as before that

$$\begin{aligned} \text{dom}_1^2 \phi_a &= F_1 \text{id}_1^0(a) \\ \text{cod}_1^2 \phi_a &= \text{id}_1^0(F_1 a). \end{aligned}$$

It remains to perform the straightforward but laborious task of verifying that these coherence data define the required natural transformations and that these natural transformations satisfy the required axioms. This in turn amounts to showing that certain diagrams of 2-cells in Y_1 commute (see [7,12] for details.) These verifications all follow the same pattern which was outlined in the Scholium of Section 3. For the reader's convenience we shall again describe this method here.

The typical diagram of 2-cells in Y_1 which we must show is commutative involves two distinct composite 2-cells which a priori are parallel in Y_1 . To show they are in fact equal it suffices to exhibit a 3-cell of Y_1 which is a bridge from one of these composite cells to the other. For by hypothesis all cells of Y_1 above dimension 2 are identity cells.

To construct the desired bridge cells one first “disassembles” the diagram in Y_1 . Note that it consists of images under F_1 of 1-cells of X_1 together with coherence 2-cells for F_1 , and possibly the image under F_1 of the associator of X_1 and the associator for Y_2 . One “reassembles”

this diagram in Y_2 by first sending the individual 1-cells of X_1 to X_2 via ρ^X , composing them as appropriate and then sending them to Y_2 via F_2 . These cells are then reconnected by the 2-cells of Y_2 whose images under λ^Y and λ^X define the coherence cells for F_1 and the coherence data for X_1 and Y_1 .

Once the diagram has been reassembled in Y_2 one simply notes that the images under κ^Y of each of its 2-cells is an identity in Y_3 because of the definition of the coherence cells as images of bridge cells in Y_2 . It follows that the two composite 2-cells in the diagram in Y_2 are bridged in Y_2 and the image under λ^Y of this bridge cell is the desired bridge cell in X_1 . ■

8.4 Homomorphisms are omega pseudo-functors

We next turn our attention to the following question. Suppose

$$h : A \rightarrow B$$

is a homomorphism between classical bicategories. Is there an omega pseudo-functor between 2-skeletal, weak omega categories whose first coordinate is h ?

Theorem 35 *Let $h : A \rightarrow B$ be a homomorphism between classical bicategories. If h is proper then there exists an omega pseudo-functor $\mathbf{F} : \mathbf{X} \rightarrow \mathbf{Y}$ where \mathbf{X}, \mathbf{Y} are 2-skeletal, $X_1 = A$, $Y_1 = B$ and $F_1 = h$.*

The proof of this result is an unpleasantly long construction but it is entirely straight forward. We shall outline the steps so that the reader may satisfy herself that the details can be filled in as necessary.

In Section 4 we constructed a functor

$$S_C^B : \mathbf{Bicat} \rightarrow \mathbf{Weak_2Cat}.$$

This functor takes a bicategory A to a 2-skeletal, weak omega category \mathbf{X} whose X_1 component is A , whose X_2 component is the bicategory freely generated by the omega graph underlying A , and whose X_3 component is the strict 2-category freely generated by the omega graph underlying A .

1. Define $\mathbf{X} = S_C^B A$ and $\mathbf{Z} = S_C^B B$. Define $F_1 : X_1 \rightarrow Y_1$ to be h . Define $F_2 : X_2 \rightarrow Y_2$ by the equation

$$F_2 \equiv \rho^Y \circ F_1 \circ \lambda^X$$

The definition of F_3 will be left to the last step of the construction. (We cannot simply use F_1 to define F_3 because the result will not be a map of omega graphs.) The plan is to modify Z_2 by adding appropriate 2-cell equivalences and 3-cell isomorphisms and to modify Z_3 by taking a quotient by the appropriate equivalence relation so that the result is a 2-skeletal, weak omega category \mathbf{Y} with the desired properties. These properties will allow the construction of F_3 which will be an omega graph morphism to Y_3 that does not in general factor through Z_3 .

2. Define an omega magma equivalence relation (definition 24) E on Z_2 to be the smallest such relation generated by all pairs of the form:

- (a) (a, b) where $(a, b, c) \in R^Z$
- (b) $(F_2(f \circ_0^1 g), F_2 f \circ_0^1 F_2 g)$
- (c) $(F_2(id_1^0 a), id_1^0(F_2 a))$

For later purposes it is important to note that $(a, b) \in E_1 \implies a \parallel b$. This follows from the fact that E_1 is generated by parallel pairs. Moreover, if $(a, b) \in E$ and $a \parallel b$ then either $a = b$ or $(a, b) \in E_1$.

3. Construct a function $\psi : E_1 \rightarrow (Z_1)_2$ which takes a pair to its associated coherence 2-cell isomorphism in Z_1 . The main difficulty here is to guarantee that ψ is well defined. The construction of ψ proceeds as follows. Let D_1 denote the equivalence relation induced on Z_2 by the ternary relation R_1^Z . One first observes that D_1 is closed in E_1 under the operation \circ_0^1 . This last assertion follows from the coherence theorem for bicategories. Clearly D_1 also contains the diagonal of E_1 . We now observe that $E_1 - D_1$ is a 2-sided ideal in E_1 in the sense that left or right composition with an element of D_1 again yields an element of $E_1 - D_1$. This observation is justified as follows. Since Z_3 is freely generated by the same omega graph which underlies Z_2 we know that

$$(a.b) \in E_1 - D_1 \Leftrightarrow \kappa^Z a \neq \kappa^Z b.$$

Since pairs in D_1 are equalized by κ^Z and because Z_3 is freely generated we conclude that

$$(a, b) \in E_1 - D_1 \text{ and } (d, d') \in D_1 \implies \kappa^Z(a \circ_0^1 d) \neq \kappa^Z(b \circ_0^1 d')$$

and similarly for left composition.

The fact that $E_1 - D_1$ is a 2-sided ideal in E_1 allows us to separate the construction of ψ into two parts. The function

$$\psi \mid D_1 \equiv \lambda^Z \mid D_1.$$

The function $\psi \mid E_1 - D_1$ is defined by induction. Recall that Z_2 is freely generated by the omega graph underlying the bicategory B . We associate with a pair of $E_1 - D_1$ the integer which is the minimum of the word lengths of the two elements of the pair. We begin the inductive construction for $n = 1$ by defining ψ using the coherence data for h . For the inductive step, we observe that any pair (a, b) of length n can be written in at least one way as the composite of two pairs of lengths $\leq n - 1$ one of which must be in $E_1 - D_1$:

$$(a, b) = (a', b') \circ_0^1 (a'', b'')$$

Now we observe that the fact that the homomorphism h is proper and the fact that its classical coherence data for h satisfies the classical axioms means that the assignment:

$$\psi(a, b) \equiv \psi(a', b') \circ_0^2 \psi(a'', b'')$$

is independent of the choice of the decomposition of (a, b) .

4. Observe that since κ^Z is a surjection in the category of omega graphs $\kappa^Z E$ is an omega magma equivalence relation on Z_3 . Define Y_3 to be the coequalizer of this relation in the category of strict omega categories and let $\widehat{\kappa}^Z$ denote the composite of κ^Z followed by the map to the coequalizer Y_3 .
5. Observe that

$$a, b \in Z_2 \text{ and } \widehat{\kappa}^Z a = \widehat{\kappa}^Z b \text{ and } a \parallel b \implies (a, b) \in E.$$

This is true for pairs already equalized by κ^Z by definition of E and for other pairs by definition of the coequalizer and the fact that $\kappa^Z E$ is transitive.

6. Construct Y_2 , a 3-skeletal omega magma, in two stages.

In the first stage define Z'_2 to be the bicategory obtained from Z_2 by freely adjoining one 2-cell for each element (a, b) of E_1 . (Recall that Z_2 is itself a free bicategory.) The domain of this 2-cell is a and its codomain is b . This gives us a monic morphism of bicategories

$$\chi : Z_2 \rightarrow Z'_2$$

which is an isomorphism on 0- and 1-cells. Define $\lambda^{Z'}$ as the extension of λ^Z obtained by sending the formal 2-cell associated with the element (a, b) to the 2-cell $\psi(a, b)$ of Z_1 defined in step 3 and extending multiplicatively. Extend $\widehat{\kappa}^Z$ over these new 2-cells by sending them to the appropriate identity cells of Y_3 .

In the second stage add a unique 3-cell isomorphism to Z'_2 for each parallel pair of 2-cells of Z'_2 equalized by $\widehat{\kappa}^Z$. This defines a 3-skeletal omega magma Z''_2 . Then define $\lambda^{Z''}$ by extending $\lambda^{Z'}$ over Z''_2 by mapping these 3-cells to identities in Z_3 . Define $\widehat{\kappa}^{Z''}$ by extending $\widehat{\kappa}^Z$ over these new 3-cells by mapping them to identities in Y_3 .

7. Define \mathbf{Y} by setting $Y_1 = Z_1$, $Y_2 = Z''_2$, $\lambda^Y = \lambda^{Z''}$, $\rho^Y = \rho^Z$, Y_3 defined in step 4 and $\kappa^Y = \widehat{\kappa}^{Z''}$.
8. Observe that \mathbf{Y} is a 2-skeletal, weak omega category because κ^Y is a categorical Penon morphism by construction. Observe that the map of globular sets

$$F_2 : X_2 \rightarrow Z_2 \hookrightarrow Y_2$$

has the properties required of the second coordinate of an omega pseudo-functor by construction of Y_3 .

9. Finally we construct a morphism of omega graphs $F_3 : X_3 \rightarrow Y_3$ which will fill in the necessary commutative diagram. This is easily done by observing that if $a \in X_3$ we may choose any element $b \in X_2$ such that $\kappa^X b = a$ and then define $F_3 a \equiv \kappa^Y \circ F_2 b$. This is well defined for the following reason. Observe that because X_2 and X_3 are freely generated κ^X equalizes only those parallel pairs of elements that must be equalized by any omega magma morphism to any strict omega category. For any parallel pair of elements $(b, b') \in X_2$ which are equalized by κ^X we can find a parallel pair of element $(c, c') \in Y_2$ which are equalized by κ^Y and such that $(F_2 b, c), (F_2 b', c') \in \dot{E}$. This follows from the last observation and from the definitions of F_2 and E . Consequently $(F_2 b, F_2 b') \in E$ and are therefore equalized by κ^Y .

The morphism F_3 defined in this way is in fact a morphism of omega graphs by the same argument. ■

9 Omega equivalence of weak omega categories

We think it fair to assert that standard (1 dimensional) category theory is built around the notion of isomorphism between objects and the attendant concepts of universal arrow and adjunction. Two dimensional

category theory permits a coarsening of the isomorphism relation via the notion of a 1-cell equivalence between objects. This leads to more general notions of limit (weighted limits). In higher dimensions one would expect that theories of higher dimensional limits and adjunctions would depend upon suitable notions of higher dimensional equivalence which would be still coarser than the notion of equivalence in the two dimensional case.

In this section we develop two good candidates for this higher dimensional notion of equivalence between weak omega categories. The coarser one, *weak equivalence*, generalizes the ordinary notion of categorical equivalence and biequivalence between bicategories. It is akin to the notion of weak homotopy equivalence in homotopy theory and is induced by a single omega functor or omega pseudo-functor. The stronger relation, *omega equivalence*, differs from weak equivalence only in that it is defined by two omega functors or omega pseudo-functors in opposite directions. In this aspect it is similar to the notion of homotopy equivalence in homotopy theory.

Both notions of equivalence depend upon the construction of a functor

$$\Pi : Tame_Omega_Cat \rightarrow Set$$

This functor is a generalization of the functor that associates with an ordinary category its set of isomorphism classes. The category *Tame_Omega_Cat* has the same objects as *PF_Omega_Cat*, the category in which an object is a weak omega category and a morphism an omega pseudo-functor. The morphisms of *Tame_Omega_Cat* are what we shall call *tame* omega pseudo-functors. An omega pseudo-functor is tame if it preserves the cells (called internal equivalences) that are used to define the functor Π on objects. For example if \mathbf{X} is a 1-skeletal weak omega category then $\Pi\mathbf{X}$ is the set of isomorphism components of X_1 (which was shown to be an ordinary category in Section 3). If \mathbf{X} is a 2-skeletal weak omega category then $\Pi\mathbf{X}$ is the set of equivalence classes of objects of X_1 (a bicategory by Section 4) where two objects are equivalent if they are connected by a 1-cell equivalence.

The notions of weak equivalence and omega equivalence are the cornerstones of a theory of weighted limits for weak omega categories which we plan to develop in part II of this work.

9.1 Omega cliques

To define the functor Π we must first define the notion of an omega clique and then exhibit a method for constructing omega cliques.

Definition 36 *An omega clique is a bridge magma (definition 11)*

(M, R^M) such that

$$a, b \in M_j \text{ and } a \parallel b \implies \exists c \text{ and } (a, b, c) \in R_j^M.$$

Recall our convention that any two elements a, b in M_0 are parallel. Thus in an omega clique any ordered pair of 0-cells are bridged by a 1-cell and the same is true of any ordered, parallel pair of j -cells for $j \geq 1$. A 1-skeletal omega clique which is also a category is a clique in the standard sense, i.e. it is equivalent to the terminal category.

We have at hand a method for constructing omega cliques which are freely generated by omega graphs. To see this let us first recall some definitions. The category Pen_Mor (Section 5) has its objects categorical Penon morphisms (definition 13). Its morphisms are the obvious commutative squares. Now fix a strict omega category A and let $Pen_Mor(A)$ denote the subcategory in which an object is a categorical Penon morphism with codomain A and in which a morphism has as its codomain leg the identity functor of A . Let $Omega_Graph^{\rightarrow}$ denote the arrow category of $Omega_Graph$. Finally, let $Omega_Graph^{\rightarrow U^A}$ denote the subcategory in which an object is an arrow to the omega graph $U_G^C A$ and in which a morphism is a commutative square in which the codomain leg is the identity on $U_G^C A$. In Section 7.2 we constructed an adjunction

$$L_{PMA}^{GMA} \dashv U_{GMA}^{PMA}$$

between the categories $Omega_Graph^{\rightarrow U^A}$ and $Pen_Mor(A)$ in which

$$U_{GMA}^{PMA} : Penon_Mor(A) \rightarrow Omega_Graph^{\rightarrow U^A}$$

is the obvious forgetful functor.

Now let Ω denote the terminal strict omega category. Observe that $U_G^C \Omega$ is the terminal omega graph. We specialize the preceding discussion to the case $A = \Omega$.

Definition 37 *Let K be an omega graph. The **omega clique freely generated by K** is the bridge magma $Cl(K)$ that is the domain of the Penon morphism $L_{PM\Omega}^{GM\Omega}(K \rightarrow U_G^C \Omega)$.*

9.2 Internal equivalences in omega magmas

We continue laying the groundwork for the construction of the functor Π . Our next goal is to identify within any omega magma M a graded subset of cells $Eq(M)$ we shall call the *internal equivalences* of M .

Let $K(2)$ denote the discrete omega graph with exactly two cells, denoted 1, 2, in dimension 0 and in which all other cells are the higher

dimensional identities associated with the cells 1, 2. Let $Cl(2)$ denote the free omega clique which is the domain of the Penon morphism $L_{PM\Omega}^{GM\Omega}(K_2 \rightarrow U_G^C\Omega)$. The omega clique $Cl(2)$ deserves to be called the free omega clique on two objects. For $i, j = 1, 2$ define unique 1-cells $c(i, j)$ of $Cl(2)$ by the condition

$$(i, j, c(i, j)) \in R_1^{Cl(2)}.$$

Let M be an omega magma and let a, b be $(i - 1)$ -cells of M . Define the omega magma $\mathbf{M}(a, b)$ by the following conditions: $M(a, b)_j \equiv \{c \in M_{i+j} \mid \text{dom}_{i-1}^{i+j} c = a \text{ and } \text{cod}_{i-1}^{i+j} c = b\}$

The functions dom, cod and id of $M(a, b)$ are those of M suitably restricted and reindexed

The partial operations of $M(a, b)$ are those of M suitably restricted and reindexed.

We note that any omega magma homomorphism $H : M \rightarrow N$ induces homomorphisms

$$H(a, b) : M(a, b) \rightarrow N(Ha, Hb)$$

for any pair of $(i - 1)$ -cells a, b of M .

Definition 38 *Let M be an omega magma. A 1-cell f is an **elementary internal equivalence** if it is an identity cell or if there exists an omega magma homomorphism*

$$H : Cl(2) \rightarrow M$$

*such that $H(c(1, 2)) = f$. For $i \geq 2$ an i -cell f of M is an **elementary internal equivalence** if it is an identity cell or if there exists an omega magma homomorphism*

$$H : Cl(2) \rightarrow M(\text{dom}_{i-2}^i f, \text{cod}_{i-2}^i f)$$

such that $H(c(1, 2)) = f$. Denote the graded subset of M consisting of the elementary internal equivalences by $\mathbf{el}(M)$. Note that $\text{el}(M)_0$ is the empty set.

We note one obvious fact which follow immediately from the definition. If M is an ordinary category thought of as a 1-skeletal omega magma then $\text{el}(M)_1$ (the 1-cells of $\text{el}(M)$) consists of the isomorphisms of M .

Next we propose to define the concept of an internal equivalence by an induction which starts with the elementary internal equivalences. We shall need some additional notation. If M is an omega magma and $S \subset M$ a graded subset define $\overline{S} \subset M$ to be the smallest graded subset of M containing all composites of the elements of S .

Definition 39 Let M be an omega magma and $S \subset M$ a graded subset. Define $T(S) \subset M$ to be the smallest graded subset which for all $k \geq 1$ contains every k -cell $f \in M_k$ satisfying the following condition:

1. $\exists g \in M_k$ and $\text{dom}_{k-1}^k g = \text{cod}_{k-1}^k f$ and $\text{cod}_{k-1}^k g = \text{dom}_{k-1}^k f$
2. $\exists h \in S$ and $\text{dom}_k^{k+1} h = f \circ_{k-1}^k g$ and $\text{cod}_k^{k+1} h = \text{id}_k^{k-1} \text{dom}_{k-1}^k f$
3. $h' \in S$ and $\text{dom}_k^{k+1} h' = g \circ_{k-1}^k f$ and $\text{cod}_k^{k+1} h' = \text{id}_k^{k-1} \text{cod}_{k-1}^k f$

We note that if f, g are as above then g is also in $T(S)$ by the symmetry of the definition.

Definition 40 Define $S_1(M) \equiv \overline{\text{el}(M)}$ and for $i \geq 2$ define $S_i(M) \equiv \overline{S_{i-1}(M) \cup T(S_{i-1}(M))}$. A k -cell f of M is an **internal equivalence** if $f \in S_i(M)$ for some $i \geq 1$. Denote the graded subset of internal equivalences by $\mathbf{Eq}(M)$.

We record the following facts.

Proposition 41 Let $F : M \rightarrow N$ be a morphism of omega magmas. If $f \in S_i(M)$ then $Ff \in S_i(N)$.

Proof:

Since F is an omega magma morphism we have $F(\text{el}(M)) \subseteq \text{el}(N)$. The desired result follows by induction after observing that all the conditions of definition 38 are preserved by omega magma homomorphisms. ■

Proposition 42 Let $a \in \text{Eq}(M)_j$ and define $f_1 \equiv \text{dom}_{j-1}^j a$ and $f_2 \equiv \text{cod}_{j-1}^j a$. Then

$$f_1 \in \text{Eq}(M)_{j-1} \iff f_2 \in \text{Eq}(M)_{j-1}$$

Proof:

Assume $f_1 \in \text{Eq}(M)_{j-1}$. Note that since $a \in \text{Eq}(M)_j$ there is a j -cell $\alpha' \in \text{Eq}(M)_j$ with $\text{dom}_{j-1}^j \alpha' = f_2$ and $\text{cod}_{j-1}^j \alpha' = f_1$. Since $f_1 \in \text{Eq}(M)_{j-1}$ there exists $g \in M_{j-1}$ and $\beta, \beta' \in \text{Eq}(M)_j$ such that

1. $\text{dom}_{j-1}^j \beta = f_1 \circ_{j-2}^{j-1} g$
2. $\text{cod}_{j-1}^j \beta = \text{id}_{j-1}^{j-2} \text{dom}_{j-2}^{j-1} f_1$
3. $\text{dom}_{j-1}^j \beta' = \text{id}_{j-1}^{j-2} \text{dom}_{j-2}^{j-1} f_1$
4. $\text{cod}_{j-1}^j \beta' = f_1 \circ_{j-2}^{j-1} g$

Then the j -cells $(\alpha \circ_{j-1}^j id_j^{j-1} g) \circ_{j-1}^j \beta$ and $(\alpha' \circ_{j-1}^j id_j^{j-1} g) \circ_{j-1}^j \beta'$ are both in $Eq(M)_j$ and this shows that $f_2 \in Eq(M)_{j-1}$. ■

Proposition 43 *Let $\kappa : X \rightarrow Y$ be a categorical Penon morphism (definition 13). Suppose $(a.b.c) \in R_i^X$. Then $c \in el(X)_{i+1}$.*

Proof:

We prove the result for $i \geq 1$; the case $i = 0$ is the same but notationally simpler.

Since $(a, b, c) \in R_i^X$ and κ is a categorical Penon morphism we know that $a \parallel b$, that $\kappa a = \kappa b$ and that

$$\kappa c = id_{i+1}^i a = id_{i+1}^i b$$

We simplify notation by setting $u = dom_{i-1}^i a$ and $v = cod_{i-1}^i a$. Note that $\kappa u = \kappa v$. Let $C(u, v)$ denote the omega submagma of $X(u, v)$ consisting whose 0-cells are a and b and whose j -cells for $j \geq 1$ are the j -cells of $X(u, v)$ which $\kappa(u, v)$ maps to identities. Observe that the cell c of X is a 1-cell of $C(u, v)$.

We now show that $c \in Eq(X)_{i+1}$ by constructing an omega magma homomorphism

$$H : Cl(2) \rightarrow C(u, v)$$

such that $H(c(1, 2)) = \dot{c}$. By definition the restriction of $\kappa(u, v)$ to $C(u, v)$ factors through the map from the terminal strict omega category Ω to $Y(\kappa u, \kappa v)$ which has as its image in dimension 0 the cell κa . Since $\kappa(u, v)$ is itself a categorical Penon morphism the bridge magma $C(u, v)$ is an omega clique (its bridge relation is the restriction of R^X to $C(u, v)$ suitably reindexed). There is an obvious map of omega graphs sending the objects 1, 2 of K_2 to $a, b \in U_G^M C(u, v)$. By adjointness we obtain a map H which by definition is a bridge morphism of omega magmas and thus has the desired property. ■

9.3 Tame omega pseudo-functors

We next identify a special subset of $Eq(M)$ which we shall term the internal contractions of M .

Definition 44 *We say that $f \in Eq(M)_j$ is an **internal contraction** and write $f \in Contr(M)_j$ if either $dom_{j-1}^j f$ and/or $cod_{j-1}^j f$ is an identity cell. We denote the graded subset of contractions by $\mathbf{Contr}(M)$.*

Definition 45 *Let M, N be omega magmas and let $F : U_S^M M \rightarrow U_S^M N$ be a morphism of the underlying globular sets. We say that F is **tame** if $F(Contr(M) \cup el(M)) \subseteq Eq(N)$.*

Thus a morphism between the underlying globular sets of two omega magmas is tame if it sends every internal contraction and every elementary internal equivalence to an internal equivalence.

Definition 46 Let $\mathbf{F} : \mathbf{X} \rightarrow \mathbf{Y}$ be an omega pseudo-functor. We say that F is a **tame omega pseudo-functor** if $F_1(Eq(X_1)) \subseteq Eq(Y_1)$

Theorem 47 Let $\mathbf{F} : \mathbf{X} \rightarrow \mathbf{Y}$ be an omega pseudo-functor. Then \mathbf{F} is tame if and only if $F_1 : U_S^M X_1 \rightarrow U_S^M Y_1$ is tame.

Proof:

Recall definition 40 and in particular that $S_1(X_1) \equiv el(X_1)$. By hypothesis $F_1(S_1(X_1)) \subseteq Eq(Y_2)$. Now suppose $f \in S_n(X_1)_j$. Then by definition there is a $g \in S_n(X_1)_j$ and a pair of $(j+1)$ -cells $h, h' \in S_{n-1}(X_1)_{j+1}$ such that

1. $dom_j^{j+1} h = f \circ_{j-1}^j g$ and $cod_j^{j+1} h = id_j^{j-1} dom_{j-1}^j f$
2. $dom_j^{j+1} h' = g \circ_{j-1}^j f$ and $cod_j^{j+1} h' = id_j^{j-1} cod_{j-1}^j f$.

Moreover, by hypothesis we know that since h, h' are both contractions $F_1(h), F_1(h') \in S_k(Y_1)$ for some k .

Next consider the cells $\rho^X f$ and $\rho^X g$ in X_2 . Denote $F_2 \rho^X f \circ_{j-1}^j F_2 \rho^X g$ by u and $F_2(\rho^X f \circ_{j-1}^j \rho^X g)$ by v . Since \mathbf{F} is an omega pseudo-functor we know that there is a $(j+1)$ -cell w such that $(u, v, w) \in R_j^Y$. By proposition 43 we know that $w \in el(Y_2)_{j+1}$. From proposition 41 we conclude that $\lambda^Y w \in el(Y_1)_{j+1}$. Reversing the roles of f, g in the arguments of F_2 yields a cell $\lambda^Y w' \in el(Y_1)_{j+1}$.

A very similar argument applied to $\rho^X id_j^{j-1} dom_{j-1}^j f$ yields a $(j+1)$ -cells $z \in el(Y_2)_{j+1}$ whose domain is $F_2(\rho^X(id_j^{j-1} dom_{j-1}^j f))$ and whose codomain is $id_j^{j-1}(F_2(dom_{j-1}^j \rho^X f))$ while replacing f by g yields a similar cell z' .

We complete the proof by considering the $(j+1)$ -cells $(\lambda^Y w \circ_{j-1}^{j+1} F_1 h) \circ_j^{j+1} \lambda^Y z$ and $(\lambda^Y z' \circ_{j-1}^{j+1} F_1 h') \circ_j^{j+1} \lambda^Y w'$. Each of these cells is an element of $S_k(Y_1)$ and hence their composites are elements of $S_{k+1}(Y_1)$. The first of these two cells has its domain the j -cell $(F_1 f \circ_{j-1}^j F_1 g)$ and its codomain the j -cell $(id_j^{j-1} dom_{j-1}^j f)$. The second has as its domain $(F_1 g \circ_{j-1}^j F_1 f)$ and as its codomain $(id_j^{j-1} cod_{j-1}^j f)$. This shows that $F_1 f \in S_{k+2}(Y_1)$. ■

Corollary 48 Let $\mathbf{F} : \mathbf{X} \rightarrow \mathbf{Y}$ be a tame omega pseudo-functor. Then $\mathbf{F}(a, b)$ is also a tame omega pseudo-functor for any choice of $(i-1)$ -cells a, b .

The preceding theorem naturally leads one to ask which classes of omega pseudo-functors can a priori be identified as tame. By proposition 41 any omega functor \mathbf{F} is a tame omega pseudo-functor because, by definition, F_1 is a morphism of omega magmas.

Theorem 49 *Let $\mathbf{F} : \mathbf{X} \rightarrow \mathbf{Y}$ be an omega pseudo-functor. If both \mathbf{X} and \mathbf{Y} are n -skeletal then \mathbf{F} is tame.*

Proof:

We show by a downward induction on the dimension of cells in $Eq(X_1)$ that $F_1(Eq(X_1)) \subseteq Eq(Y_1)$.

Let f be an n -cell of X_1 and suppose $f \in Eq(X_1)$. Note that f is a 1-cell in $X(dom_{n-1}^n f, cod_{n-2}^n f)_1$. Now $\mathbf{X}(dom_{n-1}^n f, cod_{n-2}^n f)$ is an 1-skeletal weak omega category and hence $X(dom_{n-1}^n f, cod_{n-2}^n f)_1$ is a category by the results of Section 3. If A is a category (considered as a 1-skeletal omega magma) it follows immediately from definitions 39 and 40 that $Eq(A)_1$ is the set of isomorphisms of A . Now $F(dom_{n-1}^n f, cod_{n-2}^n f)_1$ (the first coordinate of $\mathbf{F}(dom_{n-1}^n f, cod_{n-2}^n f)$) is a functor and functors preserve isomorphisms. Thus $F_1 f \in Eq(Y_1)_n$ and thus we conclude that $F_1(Eq(X_1)_n) \subseteq Eq(Y_1)_n$.

Now suppose that $F_1(Eq(X_1)_j) \subseteq Eq(Y_1)_j$ for all $j \geq k+1$. Let $f \in Eq(X_1)_k$ and let g be the k -cell and h, h' be the $(k+1)$ -cells which must witness this property as in definition 39. As in the proof of the preceding theorem we can produce a composite $(k+1)$ -cell in Y_1 with domain $(F_1 g \circ_{j-1}^j F_1 f)$ and codomain $(id_j^{j-1} cod_{j-1}^j f)$. Two of the three $(k+1)$ -cells making up this composite are in $el(Y_1)$ and the third, $F_1 h$, is in $Eq(Y_1)_{k+1}$ by induction. Thus the composite is in $Eq(Y_1)_{k+1}$. This coupled with the same argument with h' in place of h shows that $F_1 f \in Eq(Y_1)_{k+1}$ and thus completes the inductive step. ■

Next we delineate another class of tame, omega pseudo-functors.

Theorem 50 *Let \mathbf{X} be a weak omega category, a, b, c three $(i-1)$ -cells of X_1 and h an i -cell of X_1 such that $dom_{i-1}^i h = b$ and $cod_{i-1}^i h = c$. Then $\Theta(h) : \mathbf{X}(a, b) \rightarrow \mathbf{X}(a, c)$ (Section 8.2) is a tame, omega pseudo-functor.*

Proof:

$Eq(X_1)$ is closed under composition and contains all the identities of X_1 . The result then follows from the definition of $\Theta(h)_1$ which is just composition with the various identities of h . ■

9.4 The functor Π

Let \mathbf{X} be a weak omega category. We define an equivalence relation \approx on the i -cells of X_1 by setting

$$a \approx b \iff \exists f \in Eq(X_1)_{i+1} \text{ and } dom_i^{i+1} f = a \text{ and } cod_i^{i+1} f = b$$

Define $\Pi\mathbf{X}$ to be the set of \approx equivalence classes of $(X_1)_0$, the 0-cells of X_1 .

Let *Tame_Omega_Cat* denote the category in which an object is a small, weak omega category and in which a morphism is a tame pseudo-functor. It contains *Omega_Cat* as a subcategory and, by theorem 49, it contains as a full subcategory the full subcategory of *PS_Omega_Cat* (in which objects are weak omega categories and morphisms arbitrary omega pseudo-functors) containing all n -skeletal, weak omega categories for all n .

Theorem 47 then yields:

Theorem 51 $\Pi : \textit{Tame_Omega_Cat} \rightarrow \textit{Set}$ is a functor.

9.5 Weak equivalence and omega equivalence

Let $\mathbf{F} : \mathbf{X} \rightarrow \mathbf{Y}$ be a tame omega pseudo-functor and for any pair of $(i - 1)$ -cells a, b of X_1 let

$$\mathbf{F}(a, b) : \mathbf{X}(a, b) \rightarrow \mathbf{Y}(F_1 a, F_1 b)$$

be the tame omega pseudo-functor associated with \mathbf{F}, a, b .

Definition 52 \mathbf{F} is a **weak equivalence** if $\Pi\mathbf{F}$ and $\Pi\mathbf{F}(a, b)$ are isomorphisms for all $(i - 1)$ -cells a, b and all $i \geq 1$.

Definition 53 \mathbf{F} is an **omega equivalence** if there is a tame omega pseudo-functor $\mathbf{G} : \mathbf{Y} \rightarrow \mathbf{X}$ such that

$$\Pi(\mathbf{G} \circ \mathbf{F}), \Pi(\mathbf{F} \circ \mathbf{G}), \Pi(\mathbf{G} \circ \mathbf{F})(a, b), \Pi(\mathbf{F} \circ \mathbf{G})(a, b)$$

are isomorphisms for all $(i - 1)$ -cells a, b and all $i \geq 1$.

Theorem 54 Let $\mathbf{F} : \mathbf{X} \rightarrow \mathbf{Y}$ be an omega pseudo-functor. Suppose that \mathbf{F} is a weak equivalence and that \mathbf{X}, \mathbf{Y} are 1-skeletal. Then F_1 is an equivalence of ordinary categories.

Proof:

Since \mathbf{X}, \mathbf{Y} are 1-skeletal we know that \mathbf{F} is tame and from Section 3 we know that F_1 is a functor between ordinary categories. We have already observed that in this situation $Eq(X_1)_1$ consists of the isomorphisms of the category X_1 . Thus the functor F_1 is essentially surjective since it induces a bijection on isomorphism classes. The fact that X_1 is 1-skeletal means that $Eq(X_1)_j$ consists only of identities for $j \geq 2$ so F_1 must be fully faithful. ■

Theorem 55 *Let $\mathbf{F} : \mathbf{X} \rightarrow \mathbf{Y}$ be an omega pseudo-functor. Suppose that \mathbf{F} is a weak equivalence and that \mathbf{X}, \mathbf{Y} are 2-skeletal. Then F_1 is a biequivalence between bicategories.*

Proof:

Since \mathbf{X}, \mathbf{Y} are 2-skeletal we know that \mathbf{F} is tame and from the results of Sections 4 and 8 we know that F_1 is a classical pseudo-functor between bicategories. For any pair of 0-cells a, b of X_1 the preceding theorem says that the functor

$$\mathbf{F}(a, b) : \mathbf{X}(a, b) \rightarrow \mathbf{Y}(F_1 a, F_1 b)$$

is an equivalence of ordinary categories. The fact that F_1 is essentially surjective follows from the observation that the elements of $Eq(X_1)$ are precisely the 1-cell equivalences in the ordinary, bicategorical sense. ■

We next exhibit the most common examples of omega equivalences between weak omega categories.

Proposition 56 *Let \mathbf{X} be a weak omega category and let a, b be a pair of $(i - 1)$ -cells of X_1 . Then the omega pseudo-functor*

$$\Theta(\mathbf{id}_i^{i-1} \mathbf{b}) : \mathbf{X}(a, b) \rightarrow \mathbf{X}(a, b)$$

is an omega equivalence. In fact $\Pi(\Theta(\mathbf{id}_i^{i-1} \mathbf{b}))$ is the identity on $\Pi \mathbf{X}(a, b)$.

Proof:

By theorem 50 $\Theta(\mathbf{id}_i^{i-1} \mathbf{b})$ is a tame omega pseudo-functor. That it is an omega equivalence follows immediately from propositions 43 and 41 and the fact that therefore

$$f \circ_{i-1}^i \mathbf{id}_i^{i-1} b \approx f$$

via an internal equivalence (which in fact is an elementary internal equivalence). This observation also shows that $\Pi(\Theta(\mathbf{id}_i^{i-1} \mathbf{b}))$ is the identity on $\Pi \mathbf{X}(a, b)$ ■

Proposition 57 *Let \mathbf{X} be a weak omega category and let a, b, c be three $(i - 1)$ -cells of X_1 . Let h_1, h_2 be i -cells with domain b and codomain c . Suppose $h_1 \approx h_2$. Then $\Theta(\mathbf{h}_1)$ is an omega equivalence if and only if $\Theta(\mathbf{h}_2)$ is an omega equivalence.*

Proof:

This again is a straightforward consequence of the fact that $Eq(X_1)$ contains all the identity cells of X_1 and is closed under composition. ■

Theorem 58 *Let \mathbf{X} be a weak omega category and let a, b, c be three $(i - 1)$ -cells of X_1 . Let h be an i -cell with domain b and codomain c . If $h \in Eq(X_1)$ then $\Theta(\mathbf{h})$ is an omega equivalence.*

Proof:

If $h \in Eq(X_1)$ then by definition there exists $h' \in Eq(X_1)$ such that

$$h \circ_{i-1}^i h' \approx id_i^{i-1} b$$

$$h' \circ_{i-1}^i h \approx id_i^{i-1} c$$

The desired result now follows from the preceding two propositions and the fact that Π is a functor. ■

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